Title
Essays on Low_risk Investing

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Publication Date
2014

Peer reviewed|Thesis/dissertation

Essays on Low-Risk Investing

by

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A dissertation submitted in partial satisfaction of the
requirements for the degree of
Doctor of Philosophy

in

Economics

in the

GRADUATE DIVISION
of the
UNIVERSITY OF CALIFORNIA, BERKELEY

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Spring 2014

Essays on Low-Risk Investing

Abstract

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Low-risk investing refers to a diverse collection of investment strategies that emphasize low-beta, low-volatility, low idiosyncratic risk, downside protection, or risk parity. Since the 2008 financial crisis, there has been heightened interest in low-risk investing and especially in investment strategies that apply leverage to low-risk portfolios in order to enhance expected returns.

In chapter 1, we examine the well-documented *low-beta anomaly*. We show that despite the fact that low-beta portfolios had lower volatility than the market portfolio, some low-beta portfolios had higher realized Sharpe ratios (over a 22-year horizon) than the market portfolio. This is can not happen in an efficient market, where long-run return is expected to be earned as a reward for bearing risk, *if risk is equated with volatility*. We expand the notion of risk to include higher moments of the return distribution and show that excess kurtosis can make low-beta stocks and portfolios riskier than higher beta stocks and portfolios.

In chapter 2, we show that the cumulative return to a levered strategy is determined by five elements that fit together in a simple, useful formula. A previously undocumented element is the covariance between leverage and excess return to the fully invested *source* portfolio underlying the strategy. In an empirical study of volatility-targeting strategies over the 84-year period 1929–2012, this covariance accounted for a reduction in return that substantially diminished the Sharpe ratio in all cases.

In chapter 3, we gauge the return-generating potential of four investment strategies: value weighted, 60/40 fixed mix, unlevered and levered risk parity. We have three main findings. First, even over periods lasting decades, the start and end dates of a backtest can have a material effect on results; second, transaction costs can reverse ranking, especially when leverage is employed; third, a statistically significant return premium does not guarantee outperformance over reasonable investment horizons.

Acknowledgements

I am deeply indebted to my dissertation committee members Robert Anderson, James Powell, and Bin Yu. They are all exceptional teachers, mentors, and advisors.

I am particularly grateful to my co-authors on Chapters 2 and 3, Robert Anderson and Lisa Goldberg. Bob and Lisa's dogged determination are largely responsible for bringing Chapters 2 and 3 to fruition.

I would like to thank Nick Gunther, Markus Pelger, Raymond Leung, Tessa Childers-Day, and Risk Management seminar participants at Berkeley for sitting through several of my presentations and helping me frame various ideas and concepts in a precise manner.

I would also like to thank my fellow graduate students, especially my office mates Andras Komaromi, Matt Leister, and Tarso Madeira, for treating me like any other student, despite the advanced age at which I chose to pursue a Ph.D.

I gratefully acknowledge the financial support of the Center for Risk Management Research at the University of California, Berkeley.

Chapter 1 also appears as an independent working paper under the title "Looking under the Hood: What Does Quantile Regression Tell Us About the Low-Beta Anomaly." Available at SSRN: `http://ssrn.com/abstract=2424929`.

At the time of writing, Chapter 2 had been accepted for publication in the *Financial Analysts Journal* under the title "Determinants of Levered Portfolio Performance." Available at SSRN: `http://ssrn.com/abstract=2292557`.

Chapter 3 was originally published in the *Financial Analysts Journal* (Volume 68, Number 12, 2012) under the title "Will My Risk Parity Strategy Outperform?"

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Introduction

Low-risk investing refers to a diverse collection of investment strategies that emphasize low-beta, low-volatility, low idiosyncratic risk, downside protection, or risk parity. [1] The collection of low-risk strategies includes broad asset class allocations, but it also includes narrower strategies restricted to a single asset class. Since the 2008 financial crisis, there has been heightened interest in low-risk investing and especially in investment strategies that apply leverage to low-risk portfolios in order to enhance expected returns.

This dissertation asks and answers three questions about low-risk investing:

1. Are low-beta stocks and portfolios really low-risk?
2. If leverage is applied to a low-risk portfolio, what determines the return of the levered position?
3. Will a position that applies leverage to a risk parity portfolio outperform in the face of market frictions?

In chapter 1, we investigate the *low-beta anomaly*. The Capital Asset Pricing Model (CAPM) predicts that low-beta portfolios will earn a lower rate of return than the market portfolio, and will have Sharpe ratios no greater than the market portfolio. A low-beta portfolio of risky assets with beta β is predicted to earn the same rate of return as a portfolio that invests β in the market portfolio and $1-\beta$ in the risk-free asset. Empirically, neither of these predictions has been realized. Low-beta ($\beta < 1$) portfolios have earned higher returns than their market portfolio plus risk-free asset counterparts, and they have achieved higher Sharpe ratios than the market portfolio. In the literature, this is referred to as the low-beta anomaly. Chapter 1 uses quantile regression to examine other dimensions of risk beyond beta and volatility, and finds that low-beta stocks and portfolios bear additional compensated risk in the form of excess kurtosis.

In chapter 2, we show that the cumulative return to a levered strategy is determined by five elements, and they fit together in a simple, useful formula. Looking backward, our formula

[1] The simplest example of a risk parity strategy is where security or asset class portfolio weights are set so they are inversely proportional to their volatilities, i.e., it places higher weight on securities or asset classes with lower volatility and vice-versa.

can be used to attribute performance. Looking forward, an investor can populate our formula with forecasts of the five determinants in order to generate a forecast for the cumulative return to the levered strategy. The covariance term is a novel element of our formula. We find that the covariance term makes a substantial difference in the cumulative return of a levered strategy over a long horizon. In all of our empirical examples, the covariance term turned out to be negative, subtracting between 0.64% and 4.23% from annualized return over an 84-year horizon. The aggregate impact of borrowing and trading costs, the variance drag and the covariance term tended to offset the benefit of return magnification. After describing our empirical results, we examine the covariance term from the standpoint of volatility targeting. It is well known that trading and financing costs reduce the Sharpe ratio of a levered strategy. It is less well known that, via the covariance term, dynamic leverage affects the Sharpe ratio even in the absence of market frictions.

In chapter 3, we examine the historical performance of four investment strategies over an 85-year horizon. Our study includes a market or value weighted portfolio, which is the optimal risky portfolio in the CAPM, and a 60/40 mix, which is popular with pension funds and other long horizon investors. Our study also includes two risk parity strategies: [2] one that is unlevered and another that is levered to match market volatility. Our main findings are as follows:

- *Performance depends materially on the backtesting period.* Our results are consistent with the statement that the relative performance of risk parity strategies is better in turbulent periods than in bull markets. The statement is plausible, since turbulence is often accompanied by a flight to quality, during which safer (low-risk) assets tend to increase in value. However, we do not have have sufficient data to support the statement statistically.

- *Market frictions negate the outperformance of an idealized (frictionless) levered risk parity strategy.* Our results are consistent with the empirical literature on the low-beta/low-risk anomaly. Specifically, in a frictionless setting, our low-risk strategy had higher risk-adjusted returns than our high-risk strategies. However, leverage and market frictions degrade both return and risk-adjusted return.

- *A statistically significant return premium is hard to come by, and in any case, it is far from a guarantee of outperformance over reasonable investment horizons.* The confidence intervals on the returns of an investment strategy are very wide, even with many decades of data. Thus, it is rarely possible to demonstrate with conventional statistical significance that one strategy dominates another.

[2] Risk parity can refer to several different strategies, where the general goal is to set portfolio weights so that the risk contributions of all securities or asset classes are equal. The differences are in the details of how risk contributions are measured.

Chapter 1

The Low-Beta Anomaly

If the assumptions of the Capital Asset Pricing Model (CAPM) hold, the market portfolio is a mean-variance efficient portfolio [1] of risky assets. The main prescription of the CAPM is that all investors should hold the market portfolio in combination with the risk-free asset, where the fraction of wealth to be invested in the market portfolio is determined by the investor's risk tolerance. The market portfolio can be found at the point of tangency between a straight line drawn from the risk-free rate [2] and the efficient frontier of portfolios of risky assets. Since the slope of this line is the Sharpe ratio of the market portfolio, the market portfolio has the highest attainable Sharpe ratio among all possible portfolios of risky assets. The CAPM also implies that for any risky asset with return r_i:

$$\mathbb{E}[r_i] - r_f = \beta_i(\mathbb{E}[r_m] - r_f), \tag{1.1}$$

where r_m is the return on the market portfolio, r_f is the risk-free rate, and

$$\beta_i = \frac{\mathbf{C}ov(r_i, r_m)}{\mathbf{V}ar(r_m)} \tag{1.2}$$

is the asset's beta with respect to the market portfolio. [3] Hence, in *expected return - beta space* all risky asset returns will lie on a straight line emanating from the risk-free rate, with a slope equal to the expected excess return of the market. [4] This is known as the Security Market Line (SML).

Suppose a risk-averse investor is currently holding a fully invested portfolio of risky assets and the beta of this portfolio is equal to 0.5. With respect to the market portfolio,

[1] A portfolio is mean-variance efficient when expected return (mean) can not be increased without increasing the variance of the return distribution, and conversely, when the variance of the return distribution can not be decreased without decreasing expected return.

[2] In expected return - standard deviation space.

[3] This is often called the asset's *market beta* and will simply be referred to in this chapter as *beta*, unless otherwise noted.

[4] By definition, the market portfolio has a beta equal to one.

this portfolio has a low beta, i.e., it has a beta strictly less than that of the market portfolio. Further suppose that this portfolio is mean-variance efficient, meaning that it lies on the risky asset efficient frontier. Under the CAPM this investor can improve his risk-return trade-off. Specifically, he can maintain the same expected return and lower the standard deviation of his return distribution by selling the low-beta portfolio and investing half of the proceeds in the market portfolio and the other half in the risk-free asset. This new portfolio also has a beta equal to 0.5 and has the same expected return as the original portfolio, but now has a lower standard deviation than the original portfolio. The new portfolio also has the same Sharpe ratio as the market portfolio, i.e., the highest attainable Sharpe ratio. [5]

Empirically, however, some low-beta portfolios have consistently performed much better than their de-levered market portfolio counterparts, both in terms of absolute returns and in terms of risk-adjusted returns. [6] This has been documented extensively in the literature, dating back to Black (1972) and Black, Jensen and Scholes (1972), and has come to be known as the *low-beta anomaly*.

In contrast to most, if not all, of the existing literature in this area, this chapter offers a *risk-based* explanation for the low-beta anomaly. If markets are efficient [7] then return is earned purely as compensation for bearing risk, and higher return can only be earned

[5] In the parlance of finance, the investor *de-levers* the market portfolio. Put differently, the investor invests less than 100% of his investable wealth in the market portfolio and lends the remainder at the risk-free rate. This is in contrast to an investor who borrows in order to invest more than 100% of his investable wealth in the market portfolio, which is referred to as *levering* the market portfolio. There are many other strategies this investor can use to improve the risk-return trade-off, but the focus here is on maintaining the same beta.

[6] As measured by the Sharpe ratio (SR), which is defined as

$$\mathrm{SR}_p = \frac{\mathbb{E}[r_p - r_f]}{\sigma_p}, \tag{1.3}$$

where r_p is the return of the portfolio and σ_p is the volatility (standard deviation) of the portfolio. There are two measures of risk underlying the CAPM: (1) *volatility* is a measure of the *total risk* of a portfolio, and (2) *beta* is a measure of an asset's *marginal contribution* to the total risk of a portfolio. Typically, total risk is thought of in terms of a systemic component (which is correlated with the market) and an idiosyncratic component (which is uncorrelated with the market). For a well-diversified portfolio, where all idiosyncratic risk has been diversified away, leaving only the systemic component of total risk, $\sigma_p = \beta_p \sigma_m$ where β_p is the portfolio's beta and σ_m is the volatility of the market portfolio. The phrase "risk-adjusted return" generally refers to return per unit of volatility, which is proportional to beta.

[7] The realized return of a stock can be decomposed as follows:

$$\text{realized return} = \text{normal return} + \text{abnormal return}. \tag{1.4}$$

Investors expect prices to move in such a way that they earn a positive rate of return, which is compensation for risk and time. This is often called the *normal return*. Under the *efficient markets hypothesis* (EMH), the ex ante expectation of *abnormal return* is zero. There are different forms of the EMH (weak, semi-strong, and strong) depending on the information set. The sense in which efficient is used in this chapter is to assume that abnormal returns are unpredictable, i.e., they have ex ante expectation zero, without specifying the information set.

by taking on more risk (over a fixed investment horizon). This chapter finds that low-beta portfolios are indeed *riskier* than either the market portfolio or their de-levered market portfolio benchmarks, [8] even after adjusting for theoretically expected differences in volatility. The chapter presents compelling evidence that low-beta portfolios earn "extra" return for bearing "extra" risk in the form of *excess kurtosis*. In other words, low-beta portfolios tend to have higher expected returns, higher Sharpe ratios, and higher excess kurtosis than portfolios with higher betas.

The methodological approach used in the chapter employs *quantile regression* in the paradigm of the well known Fama-Macbeth procedure for analysing the relationship between beta and the cross-section of stock returns. The use of quantile regression permits study of the entire conditional distribution of stock returns, rather than just the conditional mean (as in the standard analysis). In addition, simulation is used to reveal various moments of the conditional distribution of stock returns.

The chapter proceeds as follows. Section 1.1 documents the low-beta anomaly in our dataset. Section 1.2 reviews the existing literature concerning the low-beta anomaly, kurtosis risk, and the use of quantile regression in finance. Section 1.3 details the methodological approach, which includes quantile regression, the Fama-Macbeth procedure, and simulation. Section 1.4 documents the main results of the chapter. Finally, section 1.5 concludes.

1.1 Empirical Observations

Using CRSP stock data covering the period of January 2, 1990 through December 30, 2011, we estimated individual stock betas (please see section 1.4.1 for details) and sorted the stocks from lowest beta to highest beta. We did this at the end of each month, which means that the individual stock betas changed from month to month, and formed two low-beta portfolios: the first low-beta portfolio consisted of the first third of the sorted stocks and the second consisted of the first fifth of the sorted stocks. We then compared the month ahead capitalization-weighted returns of these portfolios to the month ahead returns of the market portfolio and de-levered market portfolios, de-levered in each month to match the capitalization-weighted portfolio betas of the two low-beta portfolios. Statistics for the resulting monthly time-series of returns for each portfolio are reported in table 1.1 and the cumulative returns over the sample period are plotted in figure 1.1.

For portfolios formed from the first third of beta sorted stocks (hereafter referred to as the *first tercile*), the average beta over all months was 0.44, the average annual excess return was 5.42%, the annual volatility was 11.27%, and the realized Sharpe ratio was 0.48. [9] The

[8]Since the CAPM predicts that a low-beta portfolio with beta β will have a lower Sharpe ratio and earn the same rate of return as a portfolio that invests β in the market portfolio and $1 - \beta$ in the risk-free asset, the latter portfolio is a natural benchmark for the low-beta portfolio. The phrase *de-levered market portfolio benchmark* refers to this benchmark.

[9]Excess returns, volatilities, and Sharpe ratios are all calculated on a monthly basis and then annualized

Table 1.1: Market Portfolio vs Low Beta Portfolios

Sample Period: 07/1990-12/2011	Average Market Beta	Geometric Excess Return	Arithmetic Excess Return	Excess Return Volatility	Realized Sharpe Ratio	CAPM Realized Beta	CAPM Realized Alpha
Market Portfolio	1.03	4.90	6.07	15.84	0.38		
First Tercile Betas:							
De-Levered Market	0.44	2.84	3.11	7.81	0.40	0.44	0.43
Capitalization-Weighted	0.44	4.88	5.42	11.27	0.48	0.57	1.97
First Quintile Betas:							
De-Levered Market	0.29	1.83	2.02	6.35	0.32	0.32	0.09
Capitalization-Weighted	0.29	4.96	5.48	11.18	0.49	0.50	2.42

Notes: The data are from CRSP for the period January 2, 1990 through December 30, 2011. Betas were estimated for each stock at the end of each month using the prior 6 months of *daily* data. For each sort, the stocks in the bottom third or fifth (by number of stocks) were capitalization weighted in order to calculate the portfolio beta and the return in the *month* following portfolio formation. To compute excess returns, the U.S. 1-month T-Bill rate was used as the risk-free rate. In each month, de-levered market portfolios were constructed to match the respective capitalization-weighted betas of the portfolios formed based on sorts. *Average Market Beta* refers to the 258-month average of these betas. Realized CAPM alphas and betas were estimated by regressing a portfolio's full-sample realized monthly excess returns on a constant and the full-sample realized monthly excess returns of the market portfolio. The table reports the point estimates from these regressions. Note that the average market beta of the market portfolio is not exactly one due to differences in the estimation universe versus the market index.

results for the de-levered market portfolios were an average beta of 0.44 (by construction), an average excess return of 3.01%, an annual volatility of 7.68%, and a realized Sharpe ratio of 0.39. The corresponding results for portfolios formed from the first fifth of beta sorted stocks (hereafter referred to as the *first quintile*) were qualitatively similar: higher average excess return, higher volatility, and higher realized Sharpe ratio, vis-à-vis their de-levered market portfolio counterparts.

For further illustration of the low-beta anomaly, we analyzed some long-only portfolios constructed in a minimum variance framework, where the objective was to build a portfolio that minimized a measure of variance subject to the constraint that the resulting portfolio beta was equal to a given target. [10] Each month, we formed portfolios with three target

by multiplying by 12, $\sqrt{12}$, and $\sqrt{12}$, respectively.

[10] For each month, the portfolio stock weights are given by

$$w^* = \text{argmin}_w \; w'\beta\sigma_M^2\beta' w + w'Dw, \;\; s.t. \; w'\beta = \beta_T, \;\; w'e = 1, \;\; w \geq 0, \tag{1.5}$$

where w is the vector of portfolio stock weights, β is the vector of betas, e is a vector of ones, σ_M^2 is the variance of the market portfolio, D is the diagonal matrix of specific variances, and β_T is the target beta of the portfolio. The specific variance for each stock in each month is the variance of the residuals from the beta estimation. There are many ways to forecast portfolio variance, the implementation used here assumes that portfolio variance can be expressed as the sum of *common factor* variance (based on the CAPM one factor model) and *specific (idiosyncratic)* variance, where the common factor and specific components are betas: the market beta (i.e., one), the beta of the first tercile portfolio, and the beta of first quintile portfolio. Statistics for the monthly time-series of returns for each portfolio are reported in table 1.2, alongside the the statistics for the de-levered market and capitalization-weighted portfolios. [11] The cumulative returns over the sample period are plotted in figure 1.2. Similar to the capitalization-weighted analogs, the low-beta minimum variance portfolios achieve higher average excess returns, higher volatility, and higher realized Sharpe ratios, vis-à-vis their de-levered market portfolio benchmarks. Statistics for the time-series returns of the target-beta 1 portfolio are also provided for reference; note that the low-beta minimum variance portfolios also have higher realized Sharpe ratios than the target-beta 1 portfolio.

The top panel of figure 1.9 plots the betas of the first tercile and first quintile portfolios. As can be seen, these betas have a high degree of variability. Hence, as a supplementary check of the effect of beta on portfolio excess returns and Sharpe ratios, we constructed additional minimum variance target-beta portfolios for a series of *constant* beta targets from 0.25 to 1.75, in increments of 0.25. The results are reported in table 1.3. The biggest takeaway from this table is that the realized Sharpe ratios appear to be a strictly *decreasing* function of beta.

The fact that the low-beta portfolios have higher volatility than the corresponding de-levered market portfolios is predicted by the CAPM, since moving from the latter to the former involves moving right from the tangent line (sometimes called the *global* efficient frontier) to the risky asset efficient frontier. What is anomalous in this setting is the fact that the low-beta portfolios have higher average excess returns, which means they lie above the SML, and they have higher realized Sharpe ratios, in contrast to the notion that the market portfolio attains the maximum possible Sharpe ratio. [12]

1.2 Literature Review

1.2.1 Low-Beta Anomaly

Recognition of the low-beta anomaly dates at least back to Black (1972) and Black, Jensen and Scholes (1972). If a stock's alpha is defined as:

$$\alpha_i = (\mathbb{E}[r_i] - r_f) - \beta_i(\mathbb{E}[r_m] - r_f), \tag{1.6}$$

independent. This implementation further assumes that idiosyncratic variance is uncorrelated across stocks, i.e., all off-diagonal elements in the D matrix are set to zero.

[11] Though the empirical results for the minimum variance target-beta portfolios may be interesting in their own right, we use them here simply to illustrate the low-beta anomaly for portfolios that have been constructed using an approach different from standard beta sorts.

[12] If leverage (in this case de-leverage) is constant, then de-levered market portfolios will have the same realized Sharpe ratio as the market portfolio itself. In this example, the amount of de-leveraging is not constant, so they are only approximately the same. See chapter 2, for a detailed explanation of this point.

Table 1.2: Market Portfolio vs Low Beta Portfolios

Sample Period: 07/1990-12/2011	Average Market Beta	Geometric Excess Return	Arithmetic Excess Return	Excess Return Volatility	Realized Sharpe Ratio	CAPM Realized Beta	CAPM Realized Alpha
Market Portfolio	1.03	4.90	6.07	15.84	0.38		
Minimum Variance	1.00	7.66	8.87	16.97	0.52	1.02	2.67
First Tercile Betas:							
De-Levered Market	0.44	2.84	3.11	7.81	0.40	0.44	0.43
Capitalization-Weighted	0.44	4.88	5.42	11.27	0.48	0.57	1.97
Minimum Variance	0.44	9.10	9.41	11.61	0.81	0.57	5.94
First Quintile Betas:							
De-Levered Market	0.29	1.83	2.02	6.35	0.32	0.32	0.09
Capitalization-Weighted	0.29	4.96	5.48	11.18	0.49	0.50	2.42
Minimum Variance	0.29	9.10	9.35	11.10	0.84	0.49	6.36

Notes: The data are from CRSP for the period January 2, 1990 through December 30, 2011. Betas were estimated for each stock at the end of each month using the prior 6 months of *daily* data. For each sort, the stocks in the bottom third or fifth (by number of stocks) were capitalization weighted in order to calculate the portfolio beta and the return in the *month* following portfolio formation. To compute excess returns, the U.S. 1-month T-Bill rate was used as the risk-free rate. In each month, de-levered market and minimum variance target-beta portfolios were constructed to match the respective capitalization-weighted betas of the portfolios formed based on sorts. *Average Market Beta* refers to the 258-month average of these betas. Realized CAPM alphas and betas were estimated by regressing a portfolio's full-sample realized monthly excess returns on a constant and the full-sample realized monthly excess returns of the market portfolio. The table reports the point estimates from these regressions. Note that the average market beta of the market portfolio is not exactly one due to differences in the estimation universe versus the market index.

then equation (1.1) implies that every stock's alpha should be equal to zero. Based on monthly stock data covering the period of January 1926 through March 1966, Black, Jensen, and Scholes found that stock alphas were significantly different from zero, and that stock alphas were positive for low-beta stocks and negative for high beta stocks. In other words, stock alphas systematically depended on their betas. They posited a second factor to account for this, [13] which they called the *beta factor*:

$$\mathbb{E}[r_i] = \beta_i \mathbb{E}[r_m] + (1 - \beta_i)\mathbb{E}[r_z], \tag{1.7}$$

where r_z is the return of the beta factor, which has zero covariance with r_m. Black, Jensen, and Scholes provided empirical evidence for the existence of such a factor. Black found that restrictions on riskless borrowing, [14] with no restrictions on long or short positions in risky assets, were consistent with the empirical evidence in Black, Jensen, and Scholes. Haugen and Heins (1975) also found that over the sample periods 1926-1971 and 1946-1971, the slope of the SML was not only smaller than predicted by CAPM, but was *negative*. In

[13] In addition to the single market factor of equation (1.1).

[14] One of the assumptions of the CAPM is that every investor can lend and borrow at the risk-free rate, in unlimited quantity.

Table 1.3: Minimum Variance Target Beta Portfolios

Sample Period: 07/1990-12/2011	Average Market Beta	Geometric Excess Return	Arithmetic Excess Return	Excess Return Volatility	Realized Sharpe Ratio	Realized Beta	Realized Alpha
Low Beta Portfolios:							
Target Beta = 0.25	0.25	8.18	8.38	9.99	0.84	0.44	4.47
Target Beta = 0.50	0.50	8.64	8.95	11.25	0.80	0.60	3.60
Target Beta = 0.75	0.75	8.31	8.97	13.74	0.65	0.80	1.90
Target Beta = 1.00	1.00	7.66	8.87	16.97	0.52		
High Beta Portfolios:							
Target Beta = 1.25	1.25	7.32	9.24	20.60	0.45	1.20	-1.44
Target Beta = 1.50	1.50	6.92	9.76	24.60	0.40	1.41	-2.71
Target Beta = 1.75	1.75	6.37	10.38	28.90	0.36	1.60	-3.83

Notes: Realized alpha and realized beta are estimated with respect to the target beta 1 portfolio.

regressions of stock excess returns on their betas over both periods, they estimated negative and statistically significant coefficients on beta. Since the excess return of the market portfolio is the market price of risk under CAPM, theory predicts a positive relationship between stock betas and their excess returns.

Based on a new model of excess returns and data through 1990, Fama and French (1992) found essentially no relationship between stock excess returns and their (market) betas. They proposed a three factor model for stock excess returns:

$$\mathbb{E}[r_i] - r_f = \beta_{m,i}(\mathbb{E}[r_m] - r_f) + \beta_{h,i}\mathbb{E}[HML] + \beta_{s,i}\mathbb{E}[SMB], \tag{1.8}$$

where HML is the return to a portfolio that is long high book-to-market stocks (so-called *value* stocks) and short low book-to market stocks (so-called *growth* stocks) and SMB is the return to a portfolio that is long small stocks (in terms of market capitalization) and short big stocks. When they regressed stock excess returns on their market, value, and size betas, they found the coefficient on market beta to be small and statistically insignificant, while the coefficients for both HML and SMB were positive and statistically significant. A significant theoretical difficulty with these results, however, has been the inability to identify what sources of risk are represented by HML and SMB. Carhart (1997) added a momentum factor [15] to the Fama-French model and found evidence that this factor helped to further "explain" the cross-section of stock excess returns. Again, however, the source of risk represented by the momentum factor remains unresolved.

More recent papers have focused on explanations of the low-beta anomaly. These explanations can be broadly grouped into those based on behavioural demand and those based on limits to arbitrage. Baker, Bradley and Wurgler (2011) discuss three behavioural biases

[15] The return to the momentum factor, commonly denoted UMD, is the return to a portfolio that is long stocks that had positive returns over the previous year and short stocks that had negative returns over the previous year.

in individual investors: preference for lotteries, representativeness, and overconfidence. The preference for lotteries interpretation postulates that investors liken high-beta stocks to lottery tickets, where the magnitude of expected gains is larger than the magnitude of expected losses. This increases demand for high-beta stocks relative to low-beta stocks, increasing their prices and reducing their expected returns. Representativeness is the idea that stocks that have performed extremely well are highlighted in the financial news media. These stocks tend to have performed well relative to early stage investment, when they were more speculative and had higher expected volatility (and generally higher betas). Many investors consider these types of stocks to be representative of their actual investment opportunities, bidding up prices and reducing expected returns. Overconfidence is the idea that investors are overly optimistic about their ability to forecast future stock returns. Further, the more uncertain the outlook for a given stock, the more optimistic investors tend to be about their own forecasts of the stock's prospects. The net result is higher demand, higher prices, and lower expected returns for high volatility stocks, with the opposite implication for low-volatility stocks.

The limits to arbitrage arguments include leverage constraints, leverage aversion, and benchmarking. Cowan and Wilderman (2011), Frazzini and Pedersen (2011), and Asness, Frazzini and Pedersen (2012) contend that high-beta stocks provide implicit leverage for those investors who are less risk averse but are unwilling or unable to borrow [16] in order to achieve higher returns through explicit leverage. Baker, Bradley and Wurgler (2011) argue that benchmarking prevents many sophisticated investors (primarily institutional investors) from taking advantage of individual investor biases and exploiting the low-beta anomaly. A typical institutional investor has a mandate to manage a portfolio against a benchmark, which is often an index that is taken to represent some broad segment of the market, without using leverage. The investors performance is measured by their realized return versus the benchmark (also called active return), their realized tracking error (also called active risk), and their information ratio, which is the ratio of the difference in return versus the benchmark to the volatility of the return difference. As a result, institutional investors do not stray too far from their benchmarks, making it less likely for them to take advantage of the low-beta anomaly. As for the behavioural explanations, relative to a CAPM equilibrium all of these elements result in higher demand, higher prices, and lower expected returns for high-beta stocks, and lower demand, lower prices, and higher expected returns for low-beta stocks.

1.2.2 Kurtosis Risk

The single factor CAPM presented in equation 1.1 is consistent with a world in which investors only consider the mean and variance (first and second moments) of the distribution of stock returns in portfolio selection. Given the empirical evidence against the single factor

[16] Asness, Frazzini and Pedersen call investors who are unwilling to borrow *leverage averse*, and Frazzini and Pedersen call those that are unable to borrow *leverage constrained*.

CAPM, the relationship between expected stock returns and higher moments of the return distribution has been a topic of research dating at least back to Kraus and Litzenberger (1976). Kraus and Litzenberger specified a model that incorporated a factor representing systematic (nondiversifiable) skewness, the third moment of the return distribution. They justified a three moment CAPM on theoretical grounds [17] and presented evidence, based on stock data from January 1936 through June 1970, that the price of skewness risk is significantly different from zero, and negative. [18] Later, Harvey and Siddique (2000) obtained similar results in a conditional setting, based on stock data from July 1963 through December 1993.

The four-moment CAPM of Fang and Lai (1997) further extended the Kraus and Litzenberger three-moment model to include a factor representing systematic kurtosis, the fourth moment of the return distribution. Based on stock data from January 1974 through December 1988, they found empirical evidence for a positive price of kurtosis risk. This says that in a portfolio setting decreases in kurtosis are preferred to increases, so that assets with positive co-kurtosis with the market portfolio are expected to have higher returns. Further, well-diversified portfolios with higher expected kurtosis will have higher expected returns than the market portfolio. Amaya, Christofferson, Jacobs and Vasquez (2011) used intra-day stock data from January 1993 through September 2008 to establish a negative relationship between realized skewness and week ahead stock returns, and a positive relationship between realized kurtosis and week ahead stock returns.

1.2.3 Quantile Regression in Finance

The use of quantile regression in empirical finance is not widespread, but is growing. Quantile regression offers a non-parameteric alternative [19] to the estimation of the conditional distribution of security returns. Taylor (1999) employed quantile regression in this fashion to estimate conditional distributions of multi-period returns. Perhaps the most natural application of quantile regression in finance is the estimation of value-at-risk (VaR), which is just a specified quantile of the conditional distribution of security returns. Chernozhukov and Umantsev (2002) used quantile regression to estimate the VaR of oil stock one-day returns as a function of a small set of independent variables including the lagged one-day returns of

[17] Kraus and Litzenberger argue that investor's with utility functions that exhibit non-increasing absolute risk aversion will have a preference for positive skewness. Utility functions with this characteristic include logarithmic, power, and negative exponential utility.

[18] Non-increasing absolute risk aversion implies that in a portfolio setting, increases in skewness are preferred to decreases. Hence, assets with negative co-skewness with the market portfolio are expected to have higher returns, just as assets with positive covariance with the market portfolio are expected to have higher returns.

[19] The standard approach is to assume some parameterized distribution and simply estimate the parameters. If the assumed distribution is Gaussian, this only requires estimation of the mean and standard deviation.

the Dow Jones Industrial average, the lagged one-day returns of the spot price of oil, and the one-day lagged returns of the stock itself. Bassett and Chen (2002) utilized quantile regression to evaluate the investment style of mutual fund investment investment managers. This was done by regressing a mutual fund's return on the returns of a variety of equity indices, such as large growth, large value, small growth, small value, etc. They investigated the way style affects the returns at places other than the expected value. Barnes and Hughes (2002) employed quantile regression in the second stage of the Fama-Macbeth procedure (see section 1.3.2) to analyze the cross-section of stock returns. They found that beta is a strong cross-sectional explanatory variable at quantiles away from the median, but the market price of beta risk is not statistically significant at or near the median. Ma and Pohlman (2008) used quantile regression in both return forecasting and portfolio construction. Along the way, they also found that factor effects were not constant across return quantiles.

1.3 Quantile Regression, Fama-Macbeth, and Simulation

1.3.1 Quantile Regression

Suppose two scalar random variables X and Y are related through the linear structural equation

$$Y = \alpha_0 + X\eta_0 + \epsilon, \tag{1.9}$$

where $\epsilon \sim_{iid} F_\epsilon$ and $\mathbb{E}[\epsilon|X] = 0$. Then the ordinary least squares (OLS) solution gives an estimate of the conditional mean function of Y and the derivative of the conditional mean function with respect to X:

$$\hat{m}(X) \equiv \hat{\mathbb{E}}[Y|X] = \hat{\alpha} + X\hat{\eta}, \quad \frac{\partial \hat{m}(X)}{\partial X} = \hat{\eta}. \tag{1.10}$$

Any random variable Y can be characterized by its cumulative distribution function (CDF)

$$F_Y(y) = \int_{-\infty}^{y} f_Y(u)du = \mathbb{P}(Y \leq y), \tag{1.11}$$

and for any $\theta \in (0, 1)$

$$\mathbb{Q}_Y(\theta) \equiv F_Y^{-1}(\theta) = \inf\{y : F_Y(y) \geq \theta\} \tag{1.12}$$

is called the θth quantile of Y i.e.,

$$\mathbb{P}\left(Y \leq \mathbb{Q}_Y(\theta)\right) = \theta. \tag{1.13}$$

Using equation (1.9) and the conditional quantile restriction $\mathbb{Q}_\epsilon(\theta|X) = \nu_{\theta,0}$, the linear quantile regression model implies

$$\mathbb{Q}_Y(\theta|X) = \alpha_{\theta,0} + X\eta_{\theta,0} + \nu_{\theta,0}, \tag{1.14}$$

for some $\theta \in (0,1)$. Imposing the normalization $\nu_{\theta,0} \equiv 0$ (i.e., folding $\nu_{\theta,0}$ into $\alpha_{\theta,0}$) gives

$$\mathbb{Q}_Y(\theta|X) = \alpha_{\theta,0} + X\eta_{\theta,0}. \tag{1.15}$$

With the further (stronger) assumption that equation (1.15) holds for all $\theta \in (0,1)$, which implies $\alpha_{\theta,0} + X\eta_{\theta,0}$ is increasing in θ with probability one, the quantile regression solutions give an estimate of the conditional quantile functions of Y and the derivative of the conditional quantile functions with respect to X:

$$\hat{q}_\theta(X) \equiv \hat{\mathbb{Q}}_Y[\theta|X] = \hat{\alpha}_\theta + X\hat{\eta}_\theta, \qquad \frac{\partial \hat{q}_\theta(X)}{\partial X} = \hat{\eta}_\theta, \qquad \forall \theta \in (0,1). \tag{1.16}$$

For example, consider the linear structural equation

$$Y = \alpha_{\theta,0} + X\eta_{\theta,0} + \epsilon, \tag{1.17}$$

where $\epsilon \sim_{iid} F_\epsilon$ and ϵ is independent of X. Then

$$\mathbb{Q}_Y(\theta|X) = \alpha_{\theta,0} + X\eta_{\theta,0} + F_\epsilon^{-1}(\theta), \tag{1.18}$$

and the quantile regression solutions $\hat{\alpha}_\theta$ and $\hat{\eta}_\theta$ estimate the population parameters $\alpha_{\theta,0} + F_\epsilon^{-1}(\theta)$ and $\eta_{\theta,0}$. In other words, the quantile regression lines will all be parallel functions of X.

As a second example, consider the linear structural equation

$$Y = X\eta_{\theta,0} + (X\gamma_{\theta,0})\epsilon, \tag{1.19}$$

where heteroscedasticity has been introduced in the error terms. Then

$$\mathbb{Q}_Y(\theta|X) = X(\eta_{\theta,0} + \gamma_{\theta,0}F_\epsilon^{-1}(\theta)), \tag{1.20}$$

and the quantile regression solutions $\hat{\eta}_\theta$ estimate the population parameters $\eta_{\theta,0}+\gamma_{\theta,0}F_\epsilon^{-1}(\theta)$. In this case, the quantile regression lines will not be parallel and the domain X may need to be restricted to satisfy the condition that $X\hat{\eta}_\theta$ be increasing in θ with probability one, i.e., restricted to those X's for which the quantile regression lines do not cross.

The quantile regression estimator can be found as the solution to the following optimization problem:

$$\hat{\kappa}_\theta = \text{argmin}_\kappa \left(\sum_{i:y_i > x_i'\kappa} \theta|y_i - x_i'\kappa| + \sum_{i:y_i < x_i'\kappa} (1-\theta)|y_i - x_i'\kappa| \right), \tag{1.21}$$

and the *weighted* quantile regression estimator can be found as the solution to the following optimization problem:

$$\hat{\kappa}_\theta = \text{argmin}_\kappa \left(\sum_{i:y_i > x_i'\kappa} \theta w_i|y_i - x_i'\kappa| + \sum_{i:y_i < x_i'\kappa} (1-\theta)w_i|y_i - x_i'\kappa| \right), \tag{1.22}$$

where $\kappa = (\alpha, \eta)$ and w_i are the weights placed on each observation. Using duality and the theory of risk envelopes and risk identifiers, [20] this can be formulated as a very simple linear program. [21]

1.3.2 Fama-Macbeth Regressions

The standard Fama-Macbeth cross-sectional regression analysis is a two-step procedure. Assume there is a data set that includes security return observations over some sample period and a set of factors that are believed to be linearly related to the security excess returns through the formula

$$R = \alpha + \beta\lambda, \tag{1.23}$$

where R is the $N \times 1$ vector of security excess returns over some time period within the sample period, β is a $N \times K$ matrix of security betas with respect to K factors, λ is the $K \times 1$ vector of the prices of risk associated with each of the factors, and α is a constant term. The goal is to estimate λ and to determine which factors have a statistically significant price of risk. The factors that pass this test are said to be *priced*.

Since the security betas are unknown at the outset, the first step is to estimate them. At some time t, the betas are estimated for each security i via the time series regression

$$R_i = c_i + R_F\beta_i + u_i, \tag{1.24}$$

using historical data through time t. Here R_F is the matrix of historical factor returns through time t, c_i is the constant term, and u_i are the error terms. One regression is run for each of the N securities. This yields an *estimate* of the matrix of security betas $\hat{\beta}$.

In the second step, a cross-sectional regression is run using the returns to all N securities over the period following t, the estimated betas, and the following equation

$$R_{t,t+1} = \alpha_{t+1} + \hat{\beta}_t\lambda_{t+1} + \epsilon_{t+1}, \tag{1.25}$$

where $R_{t,t+1}$ is the vector of security returns from time t to $t+1$. This regression yields the estimates $\hat{\alpha}_{t+1}$ (a scalar) and $\hat{\lambda}_{t+1}$ (a vector). In the standard Fama-Macbeth analysis, this two-step procedure is repeated for several cross-sections and the final estimates of alpha and beta are obtained by taking time-series averages across all of the cross-sections

$$\hat{\alpha} = \frac{1}{T}\sum_{t=0}^{T-1}\hat{\alpha}_{t+1}, \quad \hat{\lambda} = \frac{1}{T}\sum_{t=0}^{T-1}\hat{\lambda}_{t+1}, \tag{1.26}$$

[20] Developed in a series of papers by Rockafellar et al.

[21] Please see Appendix A.5 for details.

where T is the total number of cross-sections. The standard errors are then given by

$$\mathrm{se}(\hat{\alpha}) = \left(\frac{1}{T^2} \sum_{t=1}^{T-1} (\hat{\alpha}_{t+1} - \hat{\alpha})^2 \right)^{1/2} \tag{1.27}$$

$$\mathrm{se}(\hat{\lambda}) = \left(\frac{1}{T^2} \sum_{t=1}^{T-1} (\hat{\lambda}_{t+1} - \hat{\lambda})^2 \right)^{1/2} . \tag{1.28}$$

1.3.3 Fama-Macbeth with Quantile Regression

In the literature, there are many variations on the Fama-Macbeth procedure outlined above. This chapter assumes the relationship between the cross-section of security excess returns and factors can be described by equation 1.23, with one factor: the market portfolio. The chapter further assumes that the conditional quantiles of the cross-section of security excess returns can be modeled as linear functions of estimated market betas:

$$q_{t,\theta}(\beta_t) \equiv Q_{t,\theta}[R_{t+1} | \beta_t = \hat{\beta}_t] = \alpha_{t,\theta} + \lambda_\theta \beta_t, \quad \forall \theta \in (0, 1), \tag{1.29}$$

where $q_{t,\theta}(\beta_t)$ denotes the conditional quantile of the cross-section of stock excess returns for month t and quantile θ, and $\hat{\beta}_t$ is the capitalization-weighted (estimated) beta across all stocks in month t. In this equation, the unknowns are $\alpha_{t,\theta}$, for month t and quantile θ, and λ_θ, for quantile θ.

In the first step, individual security market betas are estimated for each month in the sample based on equation (1.24). The second step estimation uses weighted [22] quantile regression based on panel data (with monthly excess returns) and time fixed effects:

$$R_{i,t+1} = \alpha_{\theta,t+1} + \lambda_\theta \hat{\beta}_{i,t} + \epsilon_{i,t+1}, \tag{1.30}$$

where $R_{i,t+1}$ is the excess return of security i over the month from t to $t+1$ and $\hat{\beta}_{i,t}$ is the market beta of security i, estimated using daily data through time t. In each quantile regression (one for each quantile of interest), there are a total of N equations, where $N = \sum_{t=1}^{T} n_t$, T is the number of monthly cross sections and n_t is the number of securities in each cross section. Each regression yields estimates of alpha $\hat{\alpha}_\theta$, a vector of length T, and lambda $\hat{\lambda}_\theta$, a scalar. [23] Each regression also gives estimates of the conditional quantiles of stock excess returns in each cross section, and their derivatives with respect to $\hat{\beta}_t$:

$$\hat{q}_{t,\theta}(\beta_t) \equiv \hat{Q}_{t,\theta}[R_{t+1} | \beta_t = \hat{\beta}_t] = \hat{\alpha}_{t,\theta} + \hat{\lambda}_\theta \beta_t, \qquad \frac{\partial \hat{q}_{t,\theta}(\beta_t)}{\partial \beta_t} = \hat{\lambda}_\theta. \tag{1.31}$$

[22] Using market capitalization weights.

[23] Clearly, the standard errors of these estimates must be calculated with care – details can be found in Appendix A.3.

The primary quantities of interest in these regressions are the derivatives $\hat{\lambda}_\theta$, because they indicate the sensitivity of excess return quantiles to changes in beta. The derivatives are more commonly referred to in the finance literature as *risk premia* or the *market prices of risk*.

1.3.4 Simulation

In any given cross-section, the empirical quantiles of excess returns define an empirical CDF for excess returns. In the absence of any distributional assumptions, simulation is used to get a handle on the excess return density (PDF). As an example, consider the cross-section of excess returns for July 1999. The empirical quantiles are plotted in the first panel of figure 1.3. Both equal-weighted and capitalization-weighted quantiles are plotted, but this chapter works solely in a capitalization-weighted paradigm. This helps to counter the outsized effects that returns to (very) small stocks can have on results. As can be seen in the figure, the capitalization-weighted quantiles are less extreme in the tails, i.e., they are less sensitive to the presence of outliers. The second panel of the figure displays the histogram created by drawing 25,000 random numbers from a uniform distribution and inverting the empirical CDF (defined by the capitalization-weighted quantiles) to generate 25,000 excess returns. Since the uniform distribution is continuous and the set of empirical quantiles is discrete, [24] excess returns corresponding to random numbers that fall between quantiles are interpolated. For random numbers falling below the lowest quantile or above the highest quantile, the excess returns are extrapolated. [25] The mean of this distribution is the return of the market portfolio, where the market portfolio is defined as the capitalization-weighted portfolio of all stocks in the estimation universe.

Given a set of $\hat{\lambda}_\theta$'s, one for each quantile (θ) of interest, simulation can also be used to study the effect that changing beta has on the distribution of excess returns. For example, suppose one would like to examine the effect that reducing beta by 0.25 has on the distribution of excess returns. Starting with the empirical quantiles in a given cross-section t, new quantiles can be estimated as follows

$$\hat{q}_{t,\theta}^{NEW} = q_{t,\theta}^{EMP} - 0.25 \cdot \hat{\lambda}_\theta, \tag{1.32}$$

for each θ. This yields a new CDF, which can be simulated against as described above. The new CDF corresponds to a restricted universe of stocks, which excludes the highest beta stocks so that the capitalization-weighted beta of the remaining stocks is equal to 0.75. The mean of this distribution is the excess return of the capitalization-weighted portfolio of all

[24] We specified a set of 101 quantiles (see section 1.4.1), which was used throughout the application.

[25] We used linear interpolation and extrapolation. We experimented with cubic spline interpolation and extrapolation, as well as simply truncating at the lowest and highest quantiles (instead of extrapolating), with no effect on the final conclusions in section 1.4.2.

stocks in the restricted universe. Using the simulated excess return sample as a proxy for the true distribution, various moments and properties of the distribution can be estimated.

Given a risk-free asset, a simpler way to change beta is to either leverage or de-leverage, i.e., either borrow or lend using the risk-free asset. For example, to reduce the beta of a portfolio by 0.25, a fraction $\left(\frac{0.25}{P_\beta}\right)$, where P_β is the current portfolio beta, of the portfolio is sold and invested in the risk-free asset. In this application, what this chapter defines as the *effective* market price of risk for quantile θ, denoted by $\tilde{\lambda}_\theta$, [26] turns out to be the empirical quantile itself (as will be shown later), so that new quantiles can be generated by

$$\begin{aligned} \hat{q}_{t,\theta}^{NEW} &= q_{t,\theta}^{EMP} - 0.25 \cdot \tilde{\lambda}_\theta, && (1.33)\\ &= q_{t,\theta}^{EMP} - 0.25 \cdot q_{t,\theta}^{EMP}, && (1.34)\\ &= 0.75 \cdot q_{t,\theta}^{EMP}. && (1.35) \end{aligned}$$

As before, this new CDF can be simulated against to generate a sample from the distribution whose mean is the excess return of the de-levered market portfolio. Thus, various characteristics of this distribution can be compared to those from the excess return distribution corresponding to the restricted universe of stocks (previous paragraph) with the same capitalization-weighted beta, in order to potentially reveal differences, beyond volatility, that can account for differences in mean excess returns.

1.4 The Case for Kurtosis

1.4.1 Parameter Estimation Results

The results presented in this chapter are based on daily and monthly CRSP stock data covering the period of January 2, 1990 through December 30, 2011. In the first step of the Fama-MacBeth procedure we estimated market betas for each stock in the CRSP database and each month in the sample period June 1990 through November 2011, using the standard time-series OLS regression:

$$R_i = c_i + \beta_i R_m + u_i, \tag{1.36}$$

where $R_i \equiv r_i - r_f$ is the daily excess return (r_i is the daily total return) of security i , $R_m \equiv r_m - r_f$ is the daily excess return (r_m is the daily total return) of the CRSP value-weighted market index (including dividends), and r_f is the daily 1-month T-Bill rate. Each estimation was based on six months of trailing daily data, [27] where betas were only estimated

[26] To distinguish it from the *estimated* market price of risk for quantile θ, $\hat{\lambda}_\theta$.

[27] Using six months of daily data to estimate betas was a modeling choice. The goal was to estimate betas with a reasonable degree of accuracy, while not obscuring the well recognized fact that individual stock betas vary over time.

for stocks that had at least 18 observations [28] in each of the preceding six months. In an effort to avoid estimation issues particular to small stocks, we further restricted the data sample to the largest stocks that comprised 99% of the total stock market capitalization in any given month. Note that since new betas were estimated in each month, the individual stock betas *change* from month to month.

In the second step of the Fama-MacBeth procedure, we used quantile regression to estimate $\alpha_{t,\theta}$, for month t and quantile θ, and λ_θ, for quantile θ, based on the following model with panel data and time fixed effects:

$$R_{i,t+1} = \alpha_{\theta,t+1} + \lambda_\theta \hat{\beta}_{i,t} + \epsilon_{i,t+1}. \tag{1.37}$$

One capitalization-weighted quantile regression was run for each $\theta \in \Theta$, where

$$\Theta = \{0.005, 0.01, 0.02, 0.03, \ldots, 0.97, 0.98, 0.99, 0.995\}. \tag{1.38}$$

Given the estimates of $\hat{\alpha}_{t,\theta}$ and $\hat{\lambda}_\theta$, estimates of the conditional quantiles of excess stock returns in each cross-section were also obtained, as were their derivatives with respect to $\hat{\beta}_t$:

$$\hat{q}_{t,\theta}(\beta_t) \equiv \hat{Q}_{t,\theta}[R_{t+1}|\beta_t = \hat{\beta}_t] = \hat{\alpha}_{t,\theta} + \hat{\lambda}_\theta \beta_t, \qquad \frac{\partial \hat{q}_{t,\theta}(\beta_t)}{\partial \beta_t} = \hat{\lambda}_\theta. \tag{1.39}$$

Figure 1.4 displays the average empirical (capitalization-weighted) quantiles of the cross-section of excess stock returns along with the average estimated quantiles, where the averages were taken over the 258 months in the study period. Figure 1.5 displays the derivative estimates, $\hat{\lambda}_\theta$. Note that since the estimation is carried out with panel data and time fixed effects, this gave conditional quantile estimates for each month and each quantile, but derivative estimates for each quantile with no time dimension. Hence while the plots in figure 1.4 are averages across time, the plot in figure 1.5 is not an average, it is a direct estimate. As is clear from figure 1.5, the estimate of the risk premium for the median, $\hat{\lambda}_{0.50}$, is *negative.* [29]. Table 1.4 lists the risk premium estimation results for all quantiles and table 1.5 lists the average empirical quantiles over the sample period. Figure 1.6 plots a subset of the estimated quantile lines for two months in the sample: September 1996 and September 2006. For the same quantile θ, the corresponding lines in each panel have the same slope (i.e., the risk premium associated with that quantile), but different intercepts.

1.4.2 Simulation Results

To study the effect of market beta on stock returns, we simulated using the empirical quantiles of the cross-section of excess stock returns in each month in three ways:

[28] Only 15 observations were required for September 2001, due to stock market closings in that month.

[29] Note that the OLS estimate of the risk premium for the conditional mean is negative over this period as well. This is consistent with previous literature that also found a negative relationship between (market) beta and return.

Table 1.4: Risk Premium Estimates for $\theta \in \Theta$

θ	$\hat{\lambda}_\theta$	θ	$\hat{\lambda}_\theta$	θ	$\hat{\lambda}_\theta$	θ	$\hat{\lambda}_\theta$	θ	$\hat{\lambda}_\theta$
0.5	-7.12	21.0	-2.35	42.0	-0.73	63.0	0.63	84.0	2.44
1.0	-6.20	22.0	-2.25	43.0	-0.66	64.0	0.70	85.0	2.57
2.0	-5.35	23.0	-2.17	44.0	-0.59	65.0	0.77	86.0	2.66
3.0	-4.83	24.0	-2.09	45.0	-0.53	66.0	0.83	87.0	2.79
4.0	-4.47	25.0	-2.01	46.0	-0.45	67.0	0.90	88.0	2.92
5.0	-4.23	26.0	-1.94	47.0	-0.39	68.0	0.99	89.0	3.08
6.0	-4.02	27.0	-1.89	48.0	-0.34	69.0	1.08	90.0	3.20
7.0	-3.87	28.0	-1.83	49.0	-0.29	70.0	1.17	91.0	3.35
8.0	-3.73	29.0	-1.73	50.0	-0.23	71.0	1.25	92.0	3.55
9.0	-3.63	30.0	-1.65	51.0	-0.17	72.0	1.34	93.0	3.78
10.0	-3.47	31.0	-1.57	52.0	-0.11	73.0	1.44	94.0	4.03
11.0	-3.35	32.0	-1.48	53.0	-0.06	74.0	1.51	95.0	4.25
12.0	-3.22	33.0	-1.41	54.0	0.00	75.0	1.59	96.0	4.59
13.0	-3.15	34.0	-1.34	55.0	0.08	76.0	1.66	97.0	4.93
14.0	-3.04	35.0	-1.26	56.0	0.13	77.0	1.73	98.0	5.47
15.0	-2.93	36.0	-1.19	57.0	0.20	78.0	1.82	99.0	6.41
16.0	-2.85	37.0	-1.11	58.0	0.27	79.0	1.89	99.5	7.20
17.0	-2.73	38.0	-1.05	59.0	0.34	80.0	1.99		
18.0	-2.64	39.0	-0.97	60.0	0.39	81.0	2.10		
19.0	-2.54	40.0	-0.89	61.0	0.46	82.0	2.20		
20.0	-2.44	41.0	-0.81	62.0	0.54	83.0	2.31		

Notes: The risk premia in the table are in percent per *month*. The OLS risk premimum estimate was -0.24%. These numbers are plotted in figure 1.5.

1. We used the empirical quantiles directly and simulated against the empirical CDF of excess returns. This yielded a sample from the distribution of excess returns of all stocks in the estimation universe. The mean of this distribution is the excess return of the market portfolio.

2. We adjusted the empirical quantiles using the estimated derivative for each quantile:

$$\hat{q}_{t,\theta}^{ADJ} = q_{t,\theta}^{EMP} + \hat{\lambda}_\theta \cdot \Delta\hat{\beta}_t, \quad \forall t, \forall\theta \in \Theta, \tag{1.40}$$

where $\hat{q}_{t,\theta}^{ADJ}$ is the adjusted quantile, $q_{t,\theta}^{EMP}$ is the empirical quantile, and $\Delta\hat{\beta}_t$ is the desired change in beta. To achieve a specific target beta, $\hat{\beta}_t^{LOW}$, we set $\Delta\hat{\beta}_t = \hat{\beta}_t^{LOW} - \hat{\beta}_t$, where $\hat{\beta}_t$ is the capitalization-weighted beta of the market portfolio, i.e., one, subject to some caveats noted below. We then simulated against the CDF defined by $\hat{q}_{t,\theta}^{ADJ}$, $\forall\theta \in \Theta$. This yielded a sample from the distribution of excess returns of all stocks in a restricted universe, where the capitalization-weighted beta of the remaining stocks achieved the desired *change* in beta, $\Delta\hat{\beta}_t$. Again, the mean of this distribution is the

Table 1.5: Average Empirical Quantiles for $\theta \in \Theta$

θ	$\mu\left(\tilde{\lambda}_{t,\theta}\right)$	θ	$\mu\left(\tilde{\lambda}_{t,\theta}\right)$	θ	$\mu\left(\tilde{\lambda}_{t,\theta}\right)$	θ	$\mu\left(\tilde{\lambda}_{t,\theta}\right)$	θ	$\mu\left(\tilde{\lambda}_{t,\theta}\right)$
0.5	-25.52	21.0	-5.13	42.0	-1.00	63.0	2.61	84.0	7.54
1.0	-21.27	22.0	-4.88	43.0	-0.82	64.0	2.80	85.0	7.88
2.0	-17.27	23.0	-4.64	44.0	-0.66	65.0	2.98	86.0	8.25
3.0	-15.03	24.0	-4.39	45.0	-0.50	66.0	3.16	87.0	8.66
4.0	-13.57	25.0	-4.16	46.0	-0.34	67.0	3.35	88.0	9.10
5.0	-12.41	26.0	-3.95	47.0	-0.19	68.0	3.56	89.0	9.56
6.0	-11.48	27.0	-3.74	48.0	-0.03	69.0	3.77	90.0	10.06
7.0	-10.73	28.0	-3.53	49.0	0.15	70.0	3.97	91.0	10.59
8.0	-10.06	29.0	-3.33	50.0	0.32	71.0	4.19	92.0	11.20
9.0	-9.51	30.0	-3.15	51.0	0.48	72.0	4.40	93.0	11.90
10.0	-9.01	31.0	-2.94	52.0	0.65	73.0	4.61	94.0	12.79
11.0	-8.51	32.0	-2.75	53.0	0.83	74.0	4.85	95.0	13.78
12.0	-8.06	33.0	-2.57	54.0	1.01	75.0	5.08	96.0	15.04
13.0	-7.66	34.0	-2.40	55.0	1.19	76.0	5.30	97.0	16.66
14.0	-7.27	35.0	-2.22	56.0	1.36	77.0	5.55	98.0	19.11
15.0	-6.90	36.0	-2.04	57.0	1.53	78.0	5.79	99.0	23.70
16.0	-6.54	37.0	-1.87	58.0	1.71	79.0	6.04	99.5	28.93
17.0	-6.22	38.0	-1.70	59.0	1.88	80.0	6.32		
18.0	-5.94	39.0	-1.51	60.0	2.06	81.0	6.62		
19.0	-5.67	40.0	-1.34	61.0	2.24	82.0	6.91		
20.0	-5.40	41.0	-1.17	62.0	2.42	83.0	7.21		

Notes: The average quantiles in the table are in percent per *month*. The average monthly excess return of the market over the sample period was 0.51%. These numbers are plotted in figure 1.4.

excess return of the capitalization-weighted portfolio of all stocks in the restricted universe.

3. We multiplied the empirical quantiles by the fraction of the market portfolio beta to be maintained. Hence, in this case

$$\hat{q}_{t,\theta}^{ADJ} = \left(\frac{\hat{\beta}_t + \Delta\hat{\beta}_t}{\hat{\beta}_t}\right) \cdot q_{t,\theta}^{EMP}, \quad \forall t, \forall\theta \in \Theta. \tag{1.41}$$

This yielded a sample from the distribution of excess returns of all stock plus risk-free asset pairs, where the capitalization-weighted beta of all such pairs achieved the desired *change* in beta, $\Delta\hat{\beta}_t$. It is interesting to note what the last equation implies about the *effective* derivative in this case. Starting with the first equation and substituting using the last equation gives:

$$\left(\frac{\hat{\beta}_t + \Delta\hat{\beta}_t}{\hat{\beta}_t}\right) \cdot q_{t,\theta}^{EMP} = q_{t,\theta}^{EMP} + \tilde{\lambda}_{t,\theta} \cdot \Delta\hat{\beta}_t, \tag{1.42}$$

where $\tilde{\lambda}_{t,\theta}$ is *defined as* the effective derivative. Solving for $\tilde{\lambda}_{t,\theta}$ gives

$$\tilde{\lambda}_{t,\theta} = q_{t,\theta}^{EMP} \cdot \left(\frac{1}{\hat{\beta}_t}\right), \quad \forall t, \forall \theta \in \Theta. \tag{1.43}$$

If the estimation universe were exactly the same as the universe of stocks in the market proxy, it would be true that $\hat{\beta}_t = 1$ for all t, and the quantity in parentheses would then also be equal to one for all t. Since the estimation universe is slightly different than the market proxy, what can be asserted is that

$$\frac{1}{\hat{\beta}_t} \approx 1, \quad \forall t, \tag{1.44}$$

and

$$\tilde{\lambda}_{t,\theta} \approx q_{t,\theta}^{EMP}, \quad \forall t, \forall \theta \in \Theta. \tag{1.45}$$

The mean of this distribution is the return of the de-levered market portfolio.

Figure 1.7 displays the cumulative returns of 25,000 sample paths over the study period, using first tercile portfolio betas and first quintile portfolio betas, respectively. The mean cumulative return paths for the $\hat{\lambda}_\theta$-adjusted and $\tilde{\lambda}_{t,\theta}$-adjusted quantiles (thick black and blue, respectively), as well as the de-levered market (thick magenta), are superimposed on the single path returns to the $\hat{\lambda}_\theta$-adjusted quantiles. [30] Table 1.6 shows summary statistics for the average excess return paths over the study period. Though the statistics corresponding to the $\hat{\lambda}_\theta$-adjusted quantiles are quantitatively different from their empirical counterparts in table 1.2, they are qualitatively similar. Compared to the statistics corresponding to the $\tilde{\lambda}_\theta$-adjusted quantiles, they display higher volatility, higher mean returns, and higher Sharpe ratios. In this case, they also have the same volatility as the (simulated) market portfolio, but *higher* average excess returns.

To evaluate the supposition that low-beta stocks are being compensated for kurtosis risk, we examined the cross-sectional distributions of returns coming out of the simulations. The results are summarized in table 1.7. For each month in the study period, the relevant statistics were calculated across the 25,000 sample paths. The displayed quantities are averages across months, along with the Newey-West [31] t-statistic of the difference between results based

[30] The blue line lies entirely behind the magenta line, as it should. This just verifies that the $\tilde{\lambda}_{t,\theta}$-adjusted quantiles represent the quantiles of the de-levered market portfolio.

[31] Newey-West t-statistics correct for heteroscedasticity and autocorrelation.

Table 1.6: Market Portfolio vs Low Beta Portfolios (Simulated)

Sample Period: 07/1990-12/2011	Average Market Beta	Geometric Excess Return	Arithmetic Excess Return	Excess Return Volatility	Realized Sharpe Ratio	CAPM Realized Beta	CAPM Realized Alpha
Market Portfolio	1.03	4.79	5.96	15.84	0.38		
First Teercile Betas:							
$\tilde{\lambda}_{t,\theta}$-adjusted quantiles	0.44	2.68	2.94	7.69	0.38	0.43	0.36
$\hat{\lambda}_{\theta}$-adjusted quantiles	0.44	6.01	7.12	15.83	0.45	1.00	1.17
First Quintile Betas:							
$\tilde{\lambda}_{t,\theta}$-adjusted quantiles	0.29	1.74	1.92	6.26	0.31	0.31	0.06
$\hat{\lambda}_{\theta}$-adjusted quantiles	0.29	6.38	7.47	15.85	0.47	1.00	1.51

Notes: In each month, 25,000 samples are drawn from the CDF's defined by the empirical quantiles (market), the quantiles adjusted by the effective risk premiums ($\tilde{\lambda}_{t,\theta}$, i.e., de-levered market), and the estimated risk premiums ($\hat{\lambda}_{\theta}$, i.e., pure stock low-beta). The latter two adjustments are based on the change in beta implied by the first tercile portfolio betas (an average change in beta of -0.56) and first quintile portfolio betas (an average change in beta of -0.71) in each month. The numbers reported here are based on the time-series of averages across the 25,000 samples in each month. Realized CAPM alphas and betas are with respect to the market portolio.

on the estimated risk premia ($\hat{\lambda}_{\theta}$) and the effective risk premia ($\tilde{\lambda}_{t,\theta}$), and of the difference between results based on the estimated risk premia ($\hat{\lambda}_{\theta}$) and the market quantiles. The results can be summarized as follows:

- For quantile adjustments based on the change in beta implied by the first tercile portfolio betas:
  - The average *excess return* of low-beta stocks (column 1) is higher than that of the stocks in the market and the stock positions [32] in the de-levered market portfolios. The differences are statistically significant at the 1% level versus the stocks in the market portfolio and at the 6% level versus the stock positions in the de-levered market portfolio.
  - The average *volatility* of low-beta stocks is lower than that of the stocks in the market portfolio and higher than that of the stock positions in the de-levered market portfolio. The differences are statistically significant at the 1% level in both cases.
  - The average *Sharpe ratio* for low-beta stocks is higher than that of the stocks in the market and the stock positions in the de-levered market portfolios. The differences are statistically significant at the 1% level in both cases.
  - The average *skewness* of excess returns for low-beta stocks is higher than that of the stocks in the market and the stock positions in the de-levered market portfolios. The differences are not statistically significant in either case.

[32]Stock plus risk-free asset.

- The average *excess kurtosis* of excess returns for low-beta stocks is higher than that of the stocks in the market and the stock positions in the de-levered market portfolios. The differences are statistically significant at the 1% level in both cases.
- The average *downside volatility* for low-beta stocks is lower than that of the stocks in the market portfolio and higher than that of the stock positions in the de-levered market portfolio. The differences are statistically significant at the 1% level in both cases.
- The average *expected shortfall* (95%) of low-beta stocks is lower than that of the stocks in the market portfolio and higher than that of the stock positions in the de-levered market portfolio. The differences are statistically significant at the 1% level in both cases.

• For quantile adjustments based on the change in beta implied by the first quintile portfolio betas:

  - The average *excess return* of low-beta stocks (column 1) is higher than that of the stocks in the market and the stock positions in the de-levered market portfolios. The differences are statistically significant at the 1% level versus the stocks in the market portfolio and at the 5% level versus the stock positions in the de-levered market portfolio.
  - The average *volatility* of low-beta stocks is lower than that of the stocks in the market portfolio and higher than that of the stock positions in the de-levered market portfolio. The differences are statistically significant at the 1% level in both cases.
  - The average *Sharpe ratio* for low-beta stocks is higher than that of the stocks in the market and the stock positions in the de-levered market portfolios. The differences are statistically significant at the 1% level in both cases.
  - The average *skewness* of excess returns for low-beta stocks is higher than that of the stocks in the market and the stock positions in the de-levered market portfolios. The differences are not statistically significant in either case.
  - The average *excess kurtosis* of excess returns for low-beta stocks is higher than that of the stocks in the market and the stock positions in the de-levered market portfolios. The differences are statistically significant at the 1% level in both cases.
  - The average *downside volatility* for low-beta stocks is lower than that of the stocks in the market portfolio and higher than that of the stock positions in the de-levered market portfolio. The differences are statistically significant at the 1% level in both cases.
  - The average *expected shortfall* (95%) of low-beta stocks is lower than that of the stocks in the market portfolio and higher than that of the stock positions in the

de-levered market portfolio. The differences are statistically significant at the 1% level in both cases.

Given the empirical and simulation based evidence that low-beta stocks achieved higher Sharpe ratios than the stocks in the market portfolio (and the stock positions in any scaled version of the market portfolio), a risk based explanation of the low-beta anomaly requires that other sources of compensated risk, beyond volatility risk, be considered. In theory, the market portfolio has the highest attainable Sharpe ratio, but the Sharpe ratio is a measure of the amount of excess return earned *per unit of volatility risk*. In an efficient market, higher return can only be earned for bearing more risk. Thus if we believe that abnormal returns are unpredictable, low-beta stocks must be earning "extra" return by bearing "extra" risk of some kind. The results presented in table 1.7 show that while all measures of risk are the same or elevated for low-beta stocks versus the stock positions in de-levered market portfolios, they are the same or decreased, with the notable exception of excess kurtosis, versus the stocks in the market portfolio. In addition, the results based on the first quintile portfolio betas (i.e., a larger decrease in average beta) are all qualitatively the same and quantitatively larger in magnitude. This suggests that the extra risk for which low-beta stocks are being compensated is *kurtosis risk*.

The first tercile and first quintile portfolios are "special" portfolios in that, in any given month, they represent a single draw from the population of low-beta portfolios, and in some cases involve betas that are quite low relative to the market portfolio. For the first tercile portfolio, the minimum beta was -0.05 (October 1993), the maximum beta was 0.76 (October 2011), and the average beta was 0.44. For the first quintile portfolio, the minimum beta was -0.38 (October 1993), the maximum beta was 0.68 (November 2011), and the average beta was 0.29. The time-series of betas for these portfolios is plotted in the top panel of figure 1.9. The changes in beta used in the simulations for table 1.7 are plotted in the bottom panel of figure 1.9. Some of these changes in beta are quite dramatic and they are highly variable, especially in the early part of the sample.

To check if the simulation results are robust to constant changes in beta, we ran the simulation again using the same change in beta for each month in the sample. The new results are presented in table 1.8, which displays the output for $\Delta\beta \in \{-0.20, -0.40, -0.60, -0.80\}$. Even with a modest change in beta (first panel), the difference in statistics between distribution of excess returns of low-beta stocks and the distribution of excess returns of stocks in the market portfolio are statistically significant (at the 1% level), for all statistics except skewness. While average excess return, Sharpe ratio, and excess kurtosis *increase*, volatility, downside volatility, and expected shortfall *decrease*. Further, average excess return, Sharpe ratio, and excess kurtosis appear to be strictly decreasing functions of beta, while volatility, downside volatility, and expected shortfall appear to be strictly increasing functions of beta. Of the group of risk measures highlighted in this chapter, the only risk measure that is increasing along with the Sharpe ratio, as beta decreases, is excess kurtosis.

Though this chapter is not studying high-beta stocks in detail, the simulation results for a handful of constant positive changes in beta are presented in table 1.9, which displays the output for $\Delta\beta \in \{+0.20, +0.40, +0.60, +0.80\}$. The same pattern described in the previous paragraph continues to hold. Average excess return, Sharpe ratio, and excess kurtosis *decrease* as beta increases, while volatility, downside volatility, and expected shortfall *increase*.

1.4.3 Empirical Confirmation

As one check that these results extend to low-beta portfolios, and not just individual stocks, we examined the *realized* skewness and excess kurtosis of our minimum variance target beta portfolios. Since all of these portfolios had a constant beta [33] over the sample period, excess return moments could be estimated without the added noise of variable betas. Recall from table 1.3 that excess return volatility was an increasing function of beta, while the realized Sharpe ratio was a decreasing function of beta. The upper panel of figure 1.10 plots the realized skewness of these portfolios and the lower panel plots the realized excess kurtosis. These plots show that while realized skewness was not a monotonic function of beta, realized excess kurtosis was a *decreasing* function of beta. These results are consistent with the simulation results. [34]

1.5 Concluding Remarks

As Baker, Bradley, and Wurgler (2011) point out, beta and volatility may not be the correct measures of risk. But then to explain the low-beta anomaly in terms of risk, the task is to show that low-beta stocks and portfolios are riskier than higher beta stocks and portfolios, which does not necessarily require they be more volatile. Empirically, some low-beta portfolios have outperformed (on a risk-adjusted basis) both their CAPM benchmarks (i.e., de-levered market portfolios) *and* the market portfolio itself. These portfolios have been more volatile than their CAPM benchmarks, as expected, but have been "over" compensated for that extra volatility. In the strictest sense of the CAPM, the low-beta portfolios *should* have earned the same excess return as their CAPM benchmark, placing them on the SML. But perhaps the more striking anomaly is that their realized Sharpe ratios were higher than the market portfolio. If CAPM holds, this is impossible. Thus if markets are efficient, low-beta stocks and portfolios *must be* riskier than higher beta stocks and portfolios. This chapter has shown that kurtosis may be the missing risk.

One of the assumptions of the CAPM is that investors only consider the mean and variance of excess returns in their portfolio selection decisions. Thus, it is natural to define kurtosis relative to the normal distribution, which is completely characterized by its mean

[33] The constituent stocks did change from month to month, however.
[34] This is an area of ongoing research.

and variance. With respect to the normal distribution, a distribution with the same mean and variance, and positive excess kurtosis, will have heavier tails and a higher peak. [35] Figures 1.11 and 1.12 display the density functions of excess returns of low-beta stocks, the stock positions in the de-levered market portfolio, and the stocks in the market portfolio versus normal distributions with the same means and variances, respectively, for two individual months in the simulation. [36]

For a random variable X kurtosis is formally defined as

$$\mathrm{Kurt}[X] = \frac{\mathbb{E}[(X - \mathbb{E}[X])^4]}{\left(\mathbb{E}[(X - \mathbb{E}[X])^2]\right)^2} = \frac{\mu_4}{\sigma^4}. \tag{1.46}$$

Hence, kurtosis is scale free while many other standard risk measures, such as downside volatility and expected shortfall scale with volatility. This makes kurtosis a good candidate for the dimension along which low-beta stocks and portfolios can be riskier than higher beta stocks and portfolios, since they have lower volatilities. Further, kurtosis can be contrasted with skewness (another scale free measure), which does not appear to have any bearing on the excess returns of low-beta stocks and portfolios.

[35] The normal distribution has a kurtosis of 3. Excess kurtosis is defined as kurtosis minus 3, giving the normal distribution an excess kurtosis of zero.

[36] The capitalization-weighted beta of both the low-beta stocks and the stock positions in the de-levered market portfolio match the beta of the first tercile portfolio in the respective months.

Table 1.7: Cross-Sectional Simulation Statistics (Sort-Based Betas)

First Tercile Betas	Estimated RP	Market RP	Difference	t-Statistic
Average Excess Return	7.12	5.96	1.16	16.16
Excess Return Volatility	23.71	28.78	-5.07	-17.81
Sharpe Ratio	0.36	0.26	0.11	3.59
Skewness	6.47	6.45	0.01	0.02
Excess Kurtosis	466.44	332.97	133.46	11.83
Downside Volatility	15.60	18.49	-2.89	-17.82
Expected Shortfall (95%)	14.57	17.63	-3.06	-17.63

First Tercile Betas	Estimated RP	De-Levered RP	Difference	t-Statistic
Average Excess Return	7.12	2.94	4.18	1.91
Excess Return Volatility	23.71	11.59	12.12	9.32
Sharpe Ratio	0.36	0.26	0.11	3.49
Skewness	6.47	5.95	0.52	0.69
Excess Kurtosis	466.44	332.94	133.49	11.39
Downside Volatility	15.60	7.43	8.18	9.77
Expected Shortfall (95%)	14.57	7.05	7.52	8.72

First Quintile Betas	Estimated RP	Market RP	Difference	t-Statistic
Average Excess Return	7.47	6.00	1.47	17.21
Excess Return Volatility	22.44	28.79	-6.35	-19.08
Sharpe Ratio	0.41	0.26	0.16	3.82
Skewness	6.86	6.44	0.42	0.48
Excess Kurtosis	524.48	332.62	191.86	10.08
Downside Volatility	14.86	18.50	-3.64	-18.81
Expected Shortfall (95%)	13.78	17.63	-3.85	-18.78

First Quintile Betas	Estimated RP	De-Levered RP	Difference	t-Statistic
Average Excess Return	7.47	1.92	5.55	2.09
Excess Return Volatility	22.44	8.28	14.16	10.33
Sharpe Ratio	0.41	0.26	0.15	1.76
Skewness	6.86	5.83	1.03	0.56
Excess Kurtosis	524.48	333.68	190.80	10.19
Downside Volatility	14.86	5.31	9.55	10.90
Expected Shortfall (95%)	13.78	5.05	8.73	9.51

Notes: In each month, 25,000 samples are drawn from the CDF's defined by the empirical quantiles (Market RP), the quantiles adjusted by the effective risk premiums (De-Levered RP), and the estimated risk premiums (Estimated RP, i.e., pure stock low-beta). The latter two adjustments are based on the change in beta implied by the first tercile portfolio betas (an average change in beta of -0.56) and first quintile portfolio betas (an average change in beta of -0.71) in each month. Expected shortfall (ES) for a random variable X is defined as $\mathrm{ES}_\alpha(X) \equiv -\mathbb{E}[X|X \leq \mathbb{Q}_X(1-\alpha)]$. The reported numbers are the time-series means of the cross-sectional statistics. The t-Statistics are Newey-West t-Statistics for the monthly differences in the cross-sectional statistics.

Table 1.8: Cross-Sectional Simulation Statistics (Constant $\Delta\beta$)

$\Delta\beta = -0.20$	Estimated RP	Market RP	Difference	t-Statistic
Average Excess Return	6.44	5.90	0.54	8.98
Excess Return Volatility	27.00	28.79	-1.80	-100.58
Sharpe Ratio	0.29	0.25	0.04	5.05
Skewness	6.57	6.40	0.17	0.81
Excess Kurtosis	370.45	334.70	35.75	17.96
Downside Volatility	17.47	18.50	-1.03	-56.86
Expected Shortfall (95%)	16.54	17.64	-1.10	-63.84

$\Delta\beta = -0.40$	Estimated RP	Market RP	Difference	t-Statistic
Average Excess Return	6.79	5.90	0.89	15.99
Excess Return Volatility	25.23	28.79	-3.57	-204.12
Sharpe Ratio	0.33	0.25	0.08	4.44
Skewness	6.65	6.40	0.24	0.64
Excess Kurtosis	417.29	334.70	82.59	20.56
Downside Volatility	16.47	18.50	-2.03	-100.63
Expected Shortfall (95%)	15.49	17.64	-2.15	-109.77

$\Delta\beta = -0.60$	Estimated RP	Market RP	Difference	t-Statistic
Average Excess Return	7.27	5.90	1.37	24.51
Excess Return Volatility	23.50	28.79	-5.29	-258.97
Sharpe Ratio	0.39	0.25	0.14	4.51
Skewness	6.98	6.40	0.57	0.88
Excess Kurtosis	475.07	334.70	140.37	19.59
Downside Volatility	15.46	18.50	-3.05	-134.28
Expected Shortfall (95%)	14.42	17.64	-3.23	-177.45

$\Delta\beta = -0.80$	Estimated RP	Market RP	Difference	t-Statistic
Average Excess Return	7.59	5.90	1.69	34.41
Excess Return Volatility	21.68	28.79	-7.11	-297.95
Sharpe Ratio	0.45	0.25	0.20	4.49
Skewness	7.05	6.40	0.65	0.71
Excess Kurtosis	544.57	334.70	209.87	18.28
Downside Volatility	14.42	18.50	-4.08	-160.59
Expected Shortfall (95%)	13.32	17.64	-4.32	-237.10

Notes: In each month, 25,000 samples are drawn from the CDF's defined by the empirical quantiles (Market RP) and the estimated risk premiums (Estimated RP, i.e., pure stock low-beta). The latter adjustments are based on constant changes in beta as indicated in the column heading. Expected shortfall (ES) for a random variable X is defined as $\mathrm{ES}_\alpha(X) \equiv -\mathbb{E}[X|X \leq \mathbb{Q}_X(1-\alpha)]$. The reported numbers are the time-series means of the cross-sectional statistics. The t-Statistics are Newey-West t-Statistics for the monthly differences in the cross-sectional statistics.

Table 1.9: Cross-Sectional Simulation Statistics (Constant $\Delta\beta$)

$\Delta\beta = +0.20$	Estimated RP	Market RP	Difference	t-Statistic
Average Excess Return	5.46	5.90	-0.44	-7.45
Excess Return Volatility	30.58	28.79	1.78	95.57
Sharpe Ratio	0.22	0.25	-0.03	-4.25
Skewness	6.22	6.40	-0.18	-0.93
Excess Kurtosis	302.78	334.70	-31.91	-18.68
Downside Volatility	19.54	18.50	1.04	57.02
Expected Shortfall (95%)	18.73	17.64	1.08	58.35

$\Delta\beta = +0.40$	Estimated RP	Market RP	Difference	t-Statistic
Average Excess Return	4.95	5.90	-0.95	-14.62
Excess Return Volatility	32.39	28.79	3.60	207.07
Sharpe Ratio	0.19	0.25	-0.06	-4.48
Skewness	6.14	6.40	-0.26	-0.91
Excess Kurtosis	277.45	334.70	-57.25	-22.63
Downside Volatility	20.59	18.50	2.09	102.25
Expected Shortfall (95%)	19.83	17.64	2.19	106.97

$\Delta\beta = +0.60$	Estimated RP	Market RP	Difference	t-Statistic
Average Excess Return	4.63	5.90	-1.27	-20.19
Excess Return Volatility	34.18	28.79	5.38	314.88
Sharpe Ratio	0.17	0.25	-0.09	-4.42
Skewness	6.25	6.40	-0.15	-0.38
Excess Kurtosis	256.11	334.70	-78.59	-22.51
Downside Volatility	21.60	18.50	3.09	199.74
Expected Shortfall (95%)	20.88	17.64	3.24	200.34

$\Delta\beta = +0.80$	Estimated RP	Market RP	Difference	t-Statistic
Average Excess Return	4.20	5.90	-1.70	-25.46
Excess Return Volatility	35.95	28.79	7.16	343.29
Sharpe Ratio	0.14	0.25	-0.11	-4.35
Skewness	6.19	6.40	-0.22	-0.44
Excess Kurtosis	236.14	334.70	-98.55	-23.89
Downside Volatility	22.61	18.50	4.11	204.91
Expected Shortfall (95%)	21.94	17.64	4.29	223.36

Notes: In each month, 25,000 samples are drawn from the CDF's defined by the empirical quantiles (Market RP) and the estimated risk premiums (Estimated RP, i.e., pure stock low-beta). The latter adjustments are based on constant changes in beta as indicated in the column heading. Expected shortfall (ES) for a random variable X is defined as $\text{ES}_\alpha(X) \equiv -\mathbb{E}[X|X \leq \mathbb{Q}_X(1-\alpha)]$. The reported numbers are the time-series means of the cross-sectional statistics. The t-Statistics are Newey-West t-Statistics for the monthly differences in the cross-sectional statistics.

Figure 1.1: Cumulative Returns to Low-Beta Portfolios

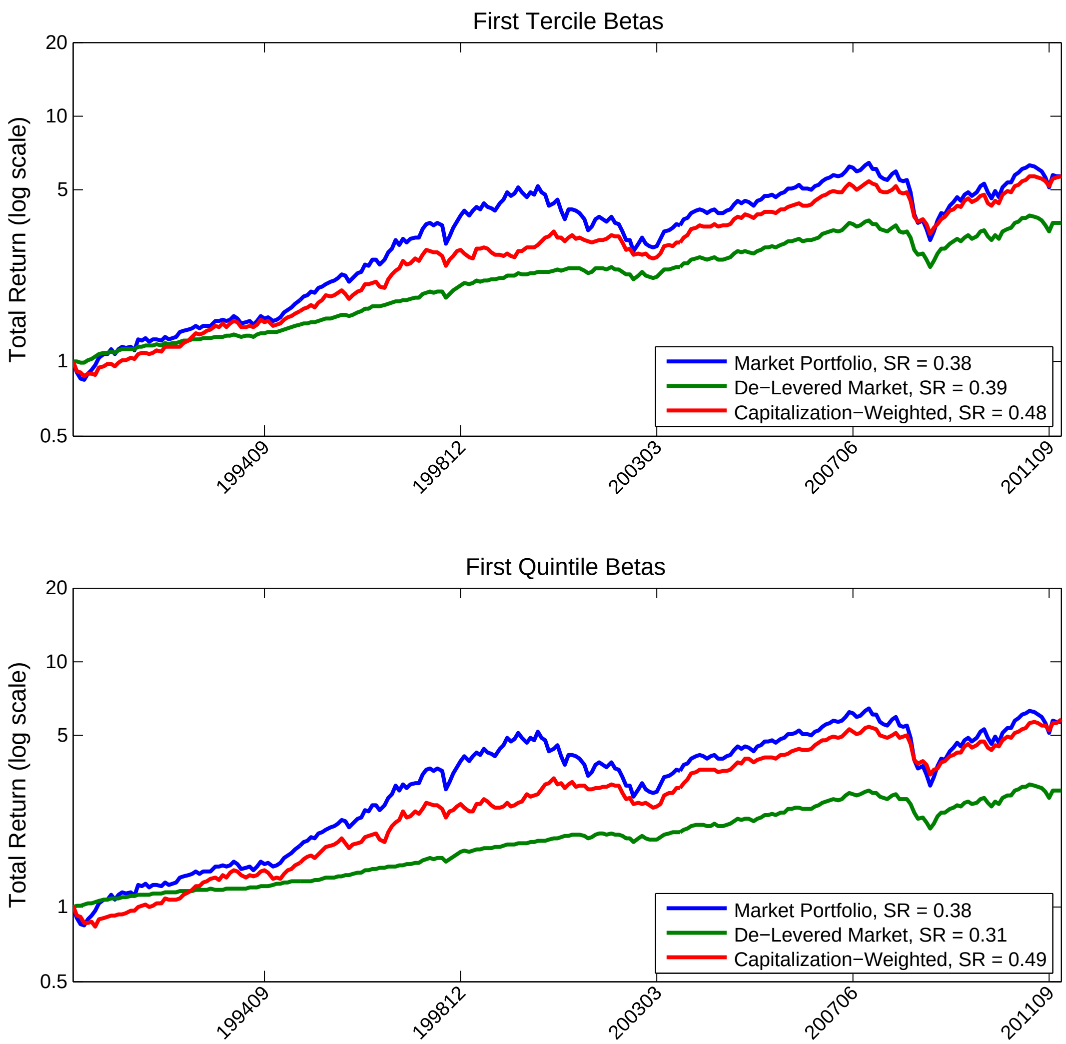


Figure 1.2: Cumulative Returns to Low-Beta Portfolios

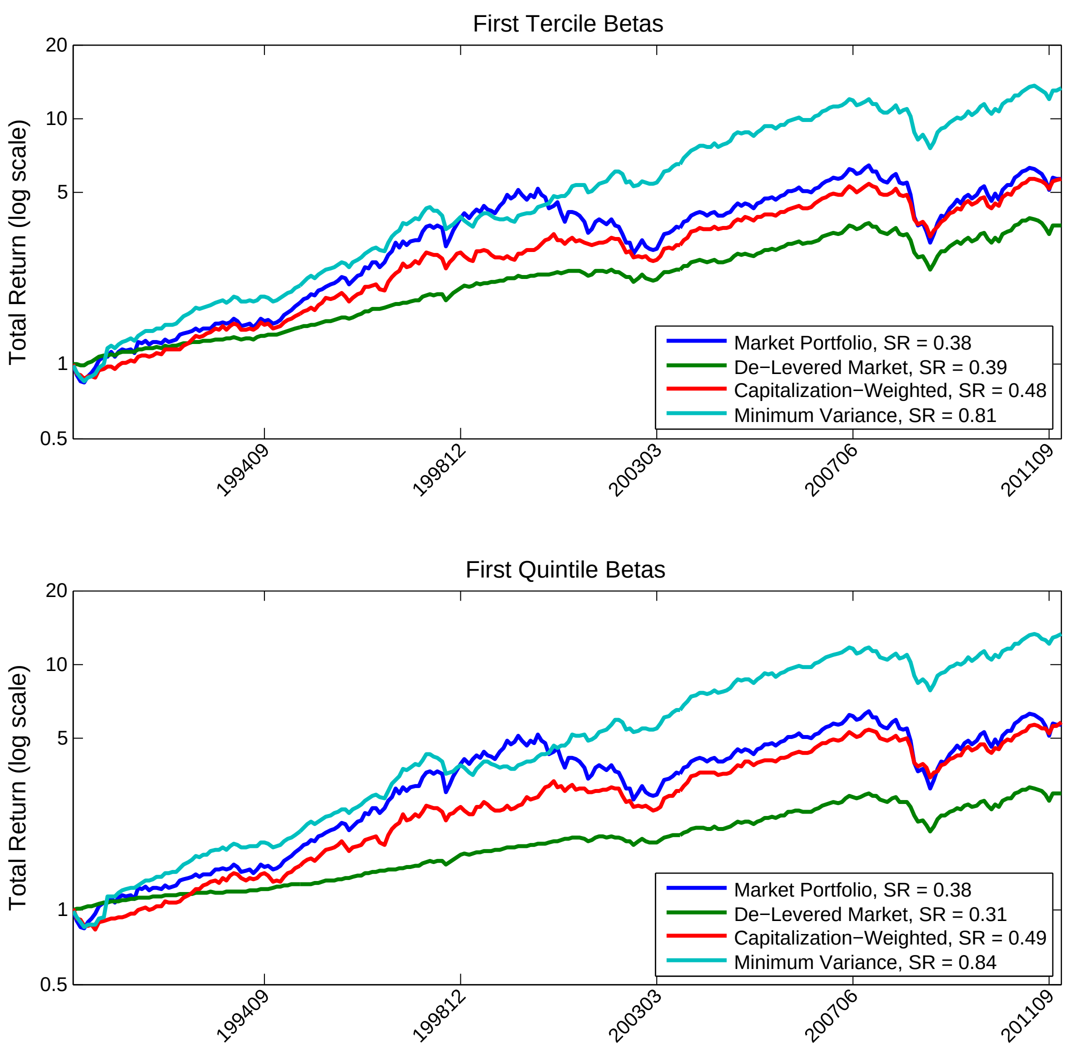


Figure 1.3: Empirical Quantiles and Simulated Excess Returns (July 1999)

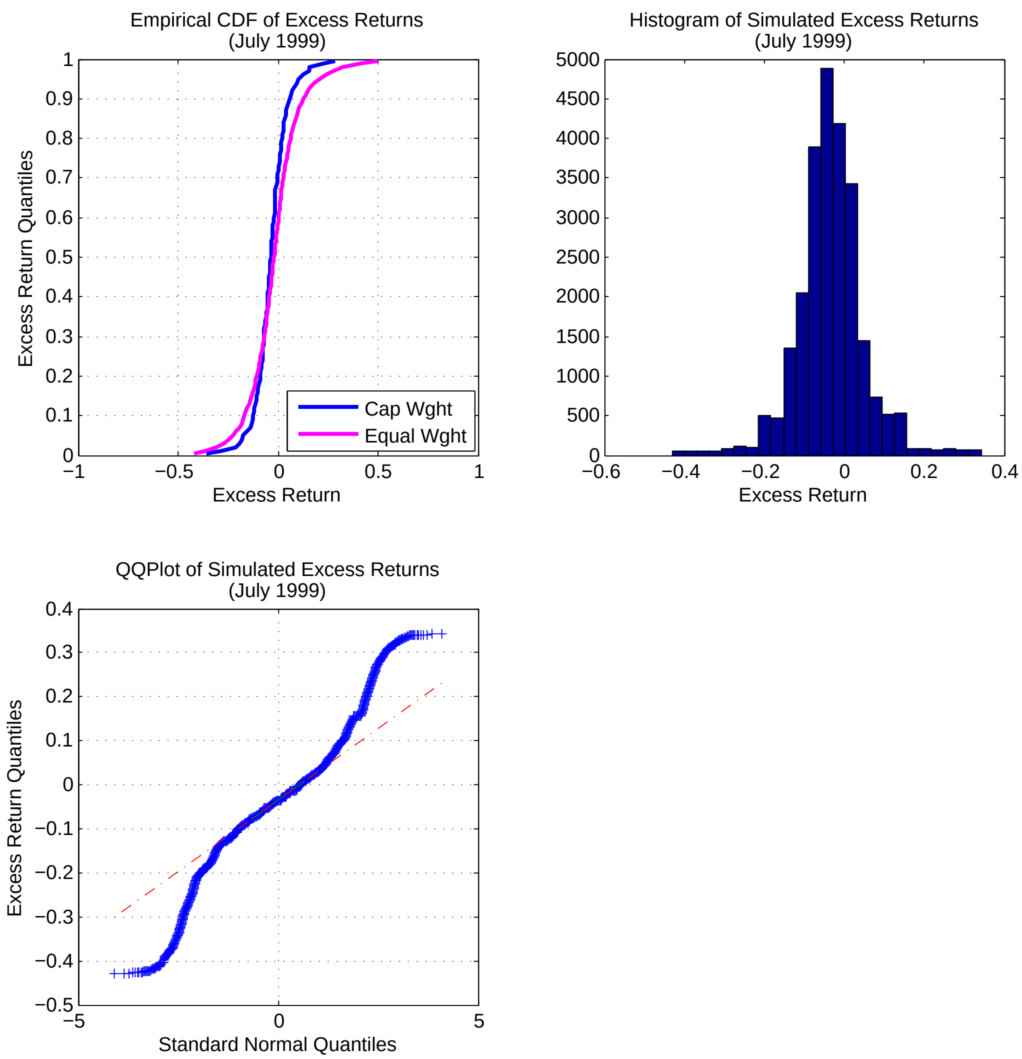


Figure 1.4: Average Empirical and Estimated Quantiles of Excess Stock Returns

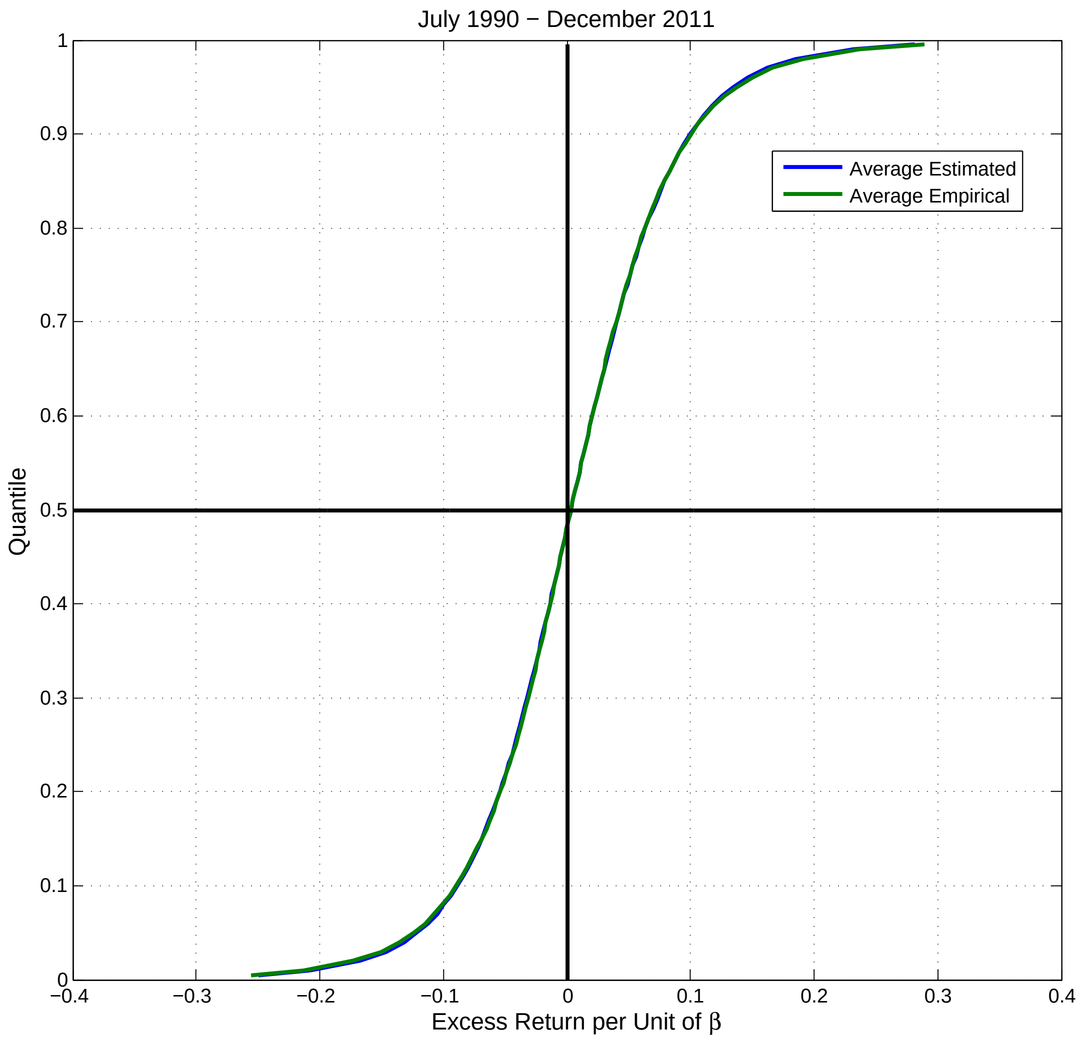


Figure 1.5: Estimates of λ_θ for $\theta \in \Theta$

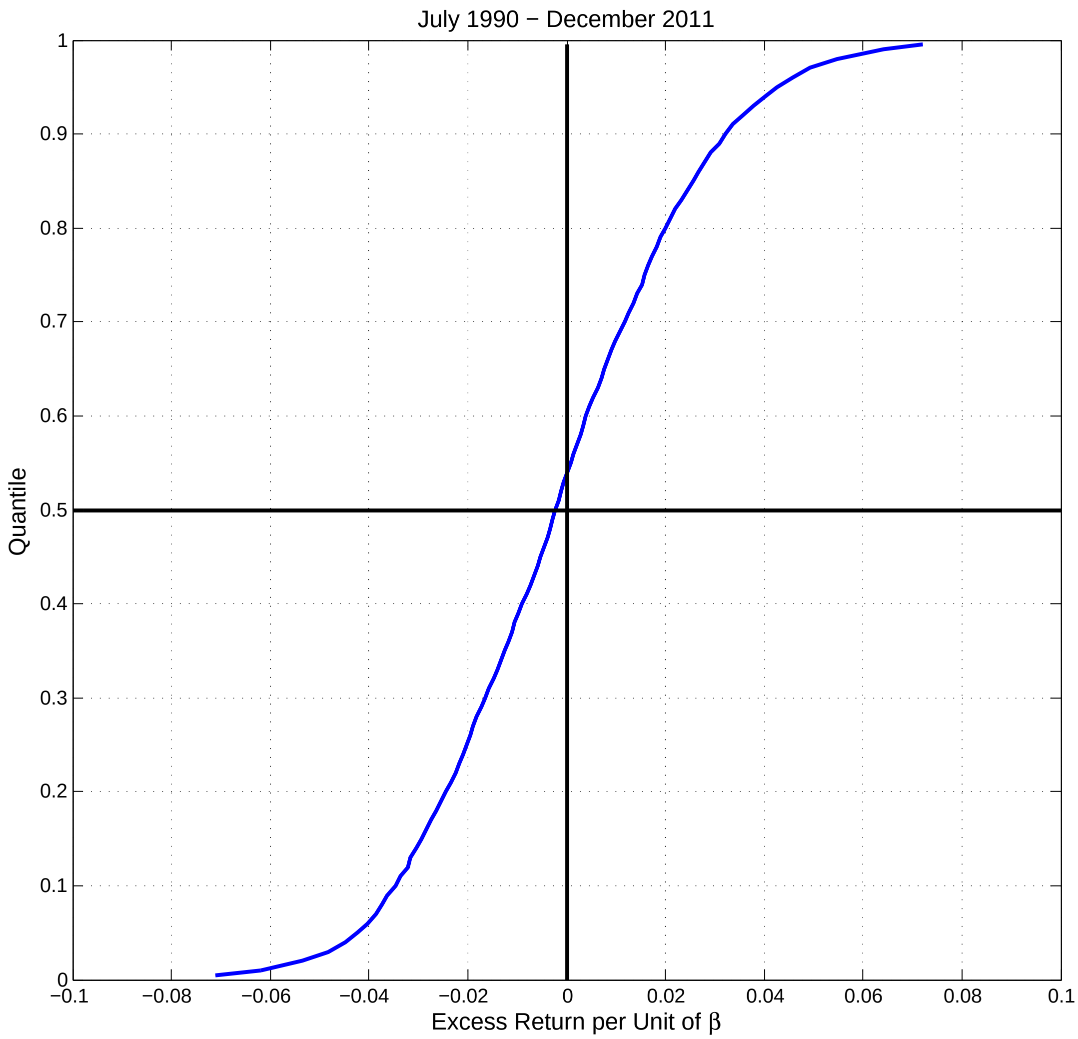


Figure 1.6: Estimated Quantile Lines for $\theta \in \{0.005, 0.05, 0.10, \ldots, 0.90, 0.95, 0.995\}$

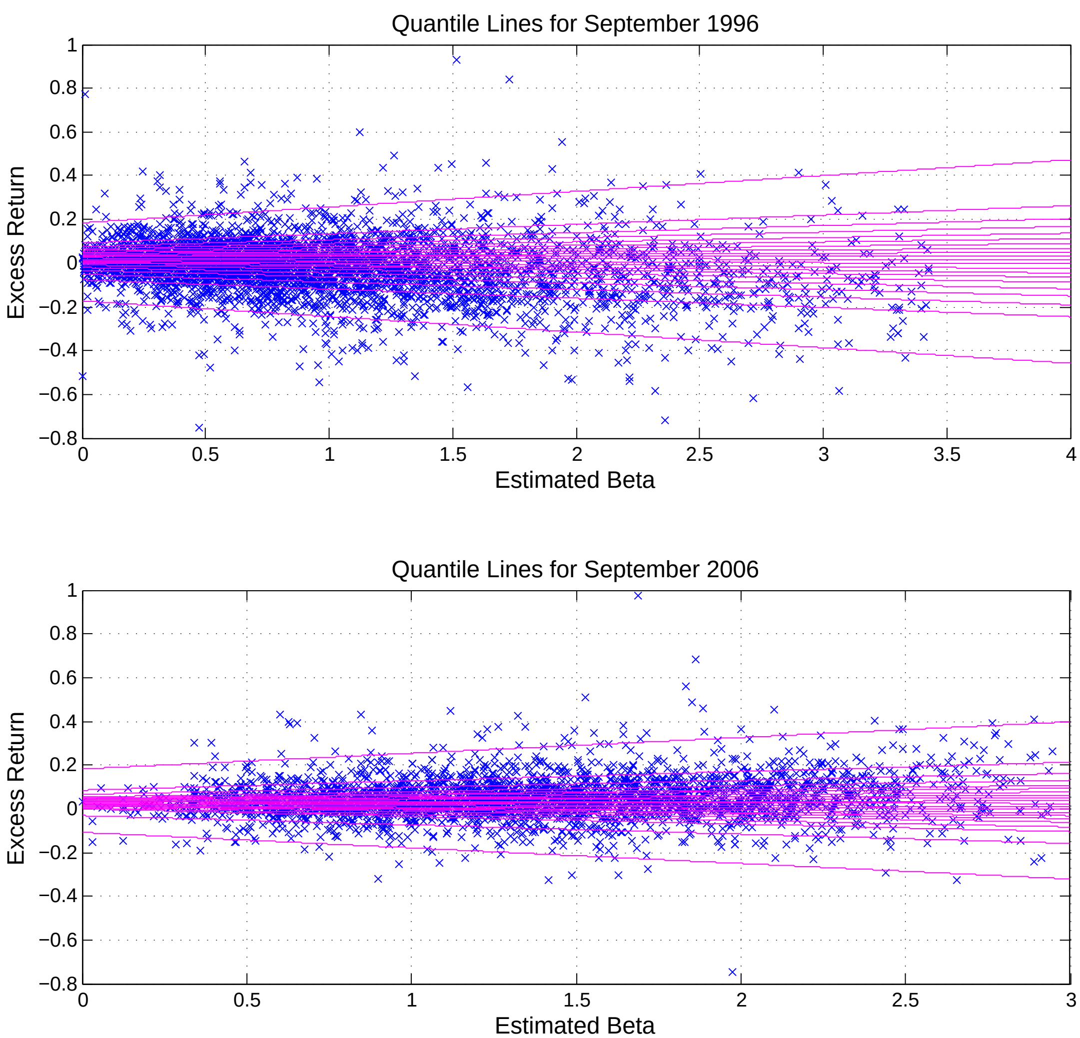


Figure 1.7: Simulated Cumulative Returns to Low-Beta Portfolios

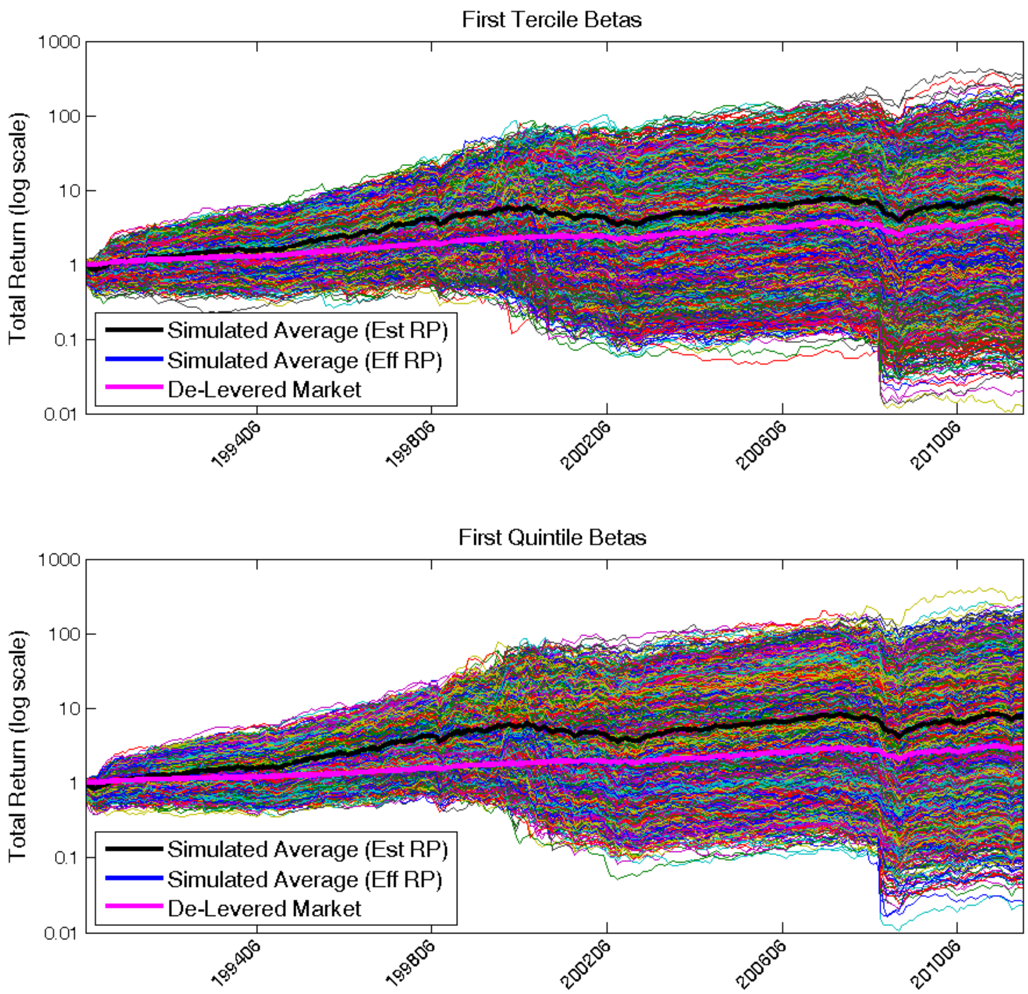


Figure 1.8: Estimated and (Average) Effective Risk Premia

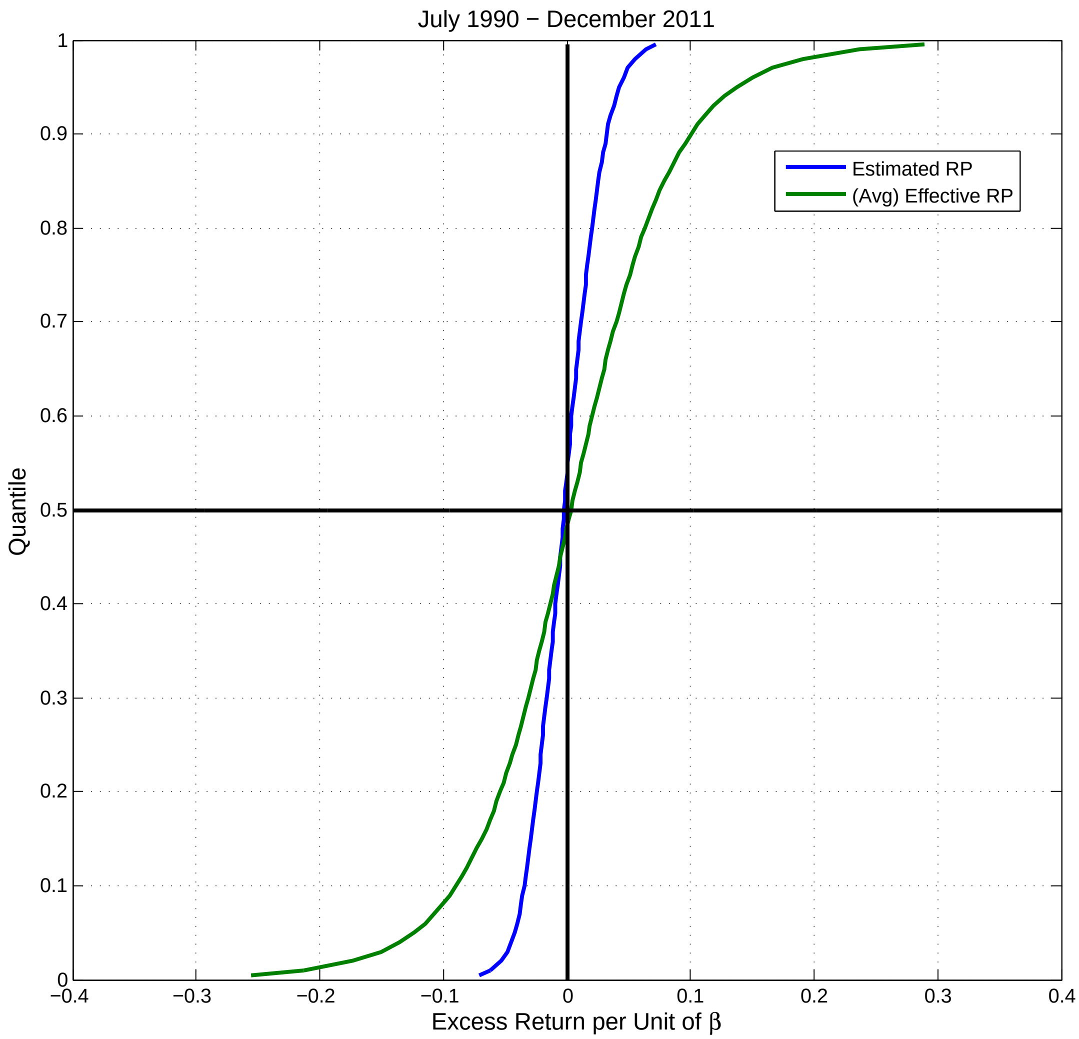
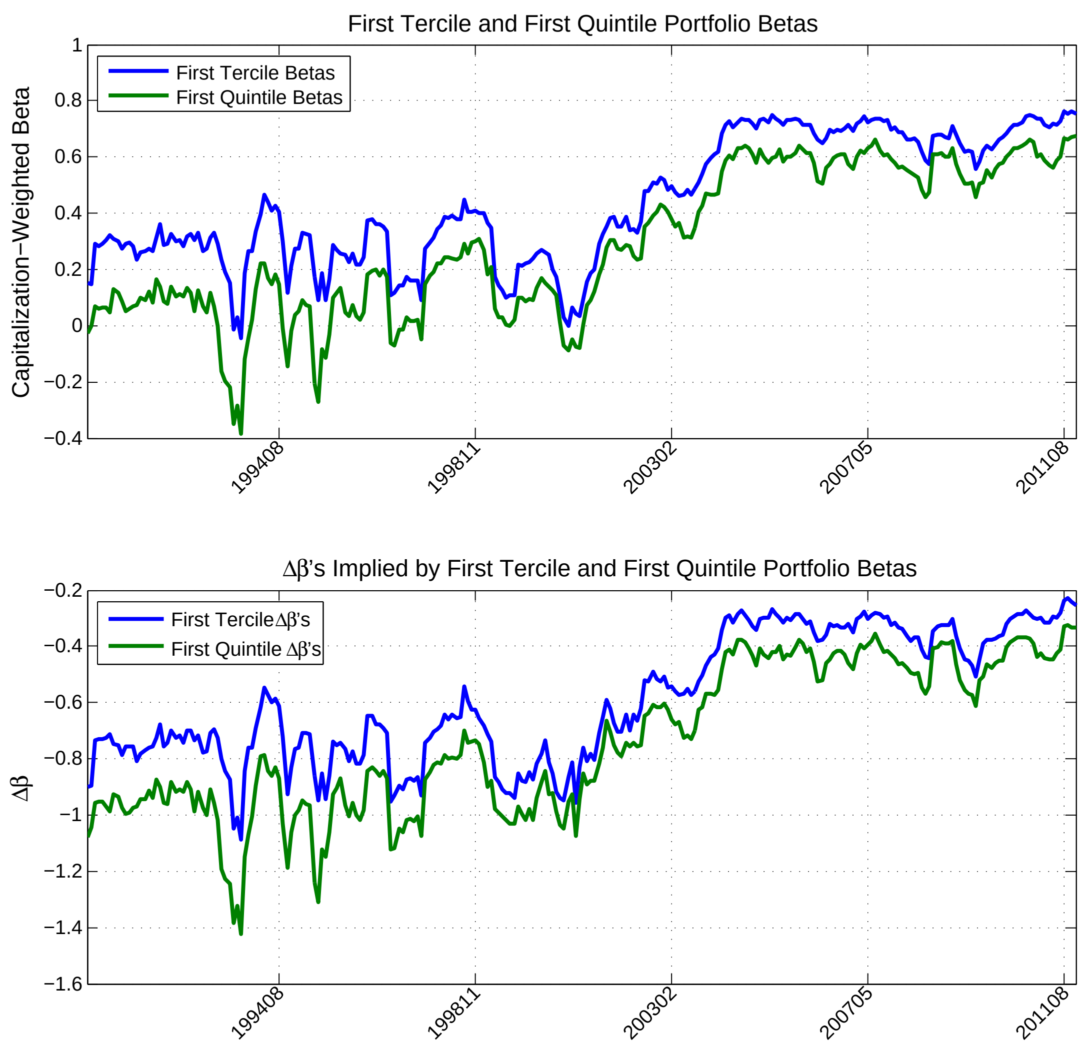


Figure 1.9: Portfolio Betas Based on Market Beta Sort in Each Month



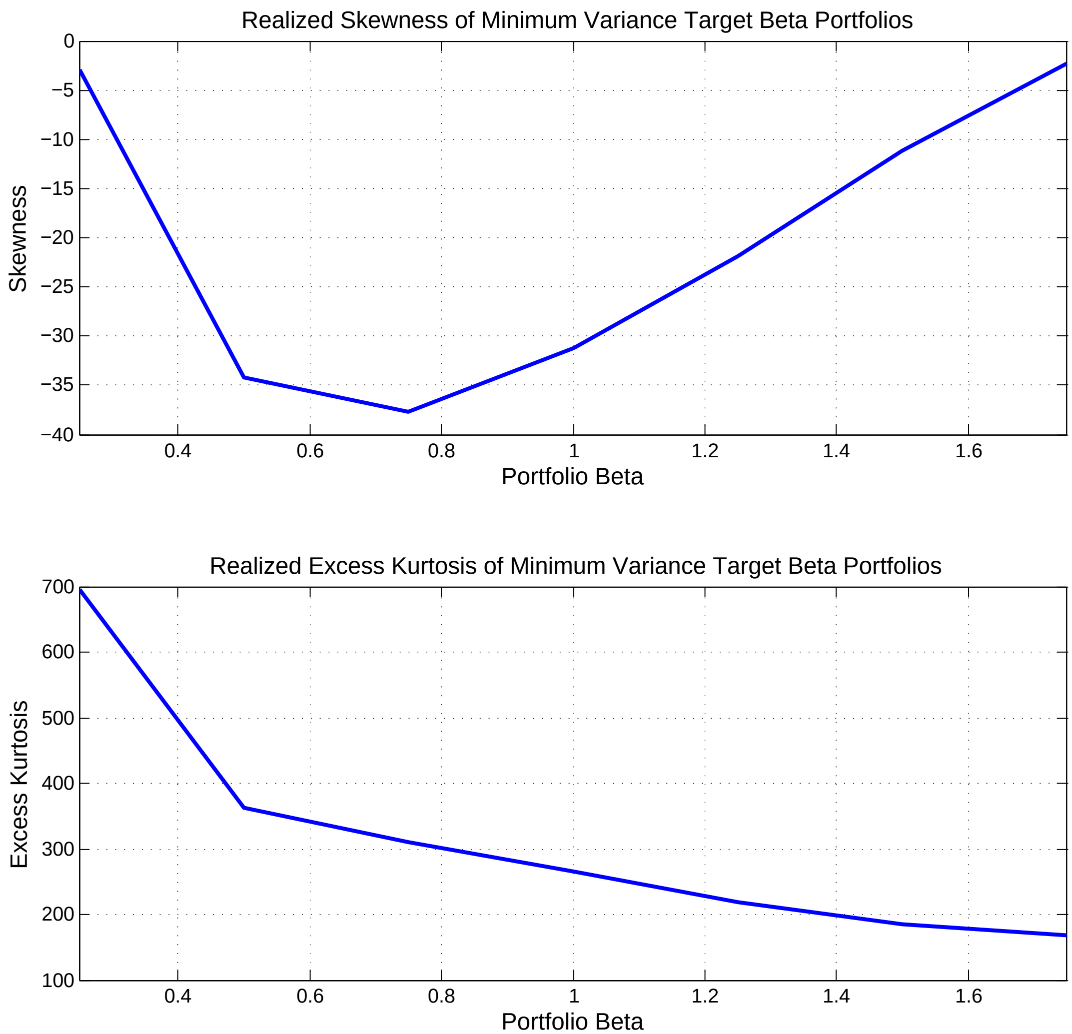


Figure 1.10: Realized Skewness and Excess Kurtosis of Minimum Variance Target Beta Portfolios

Figure 1.11: Simulated Kurtosis for Low-Beta, De-Levered Market, and Market Portfolios

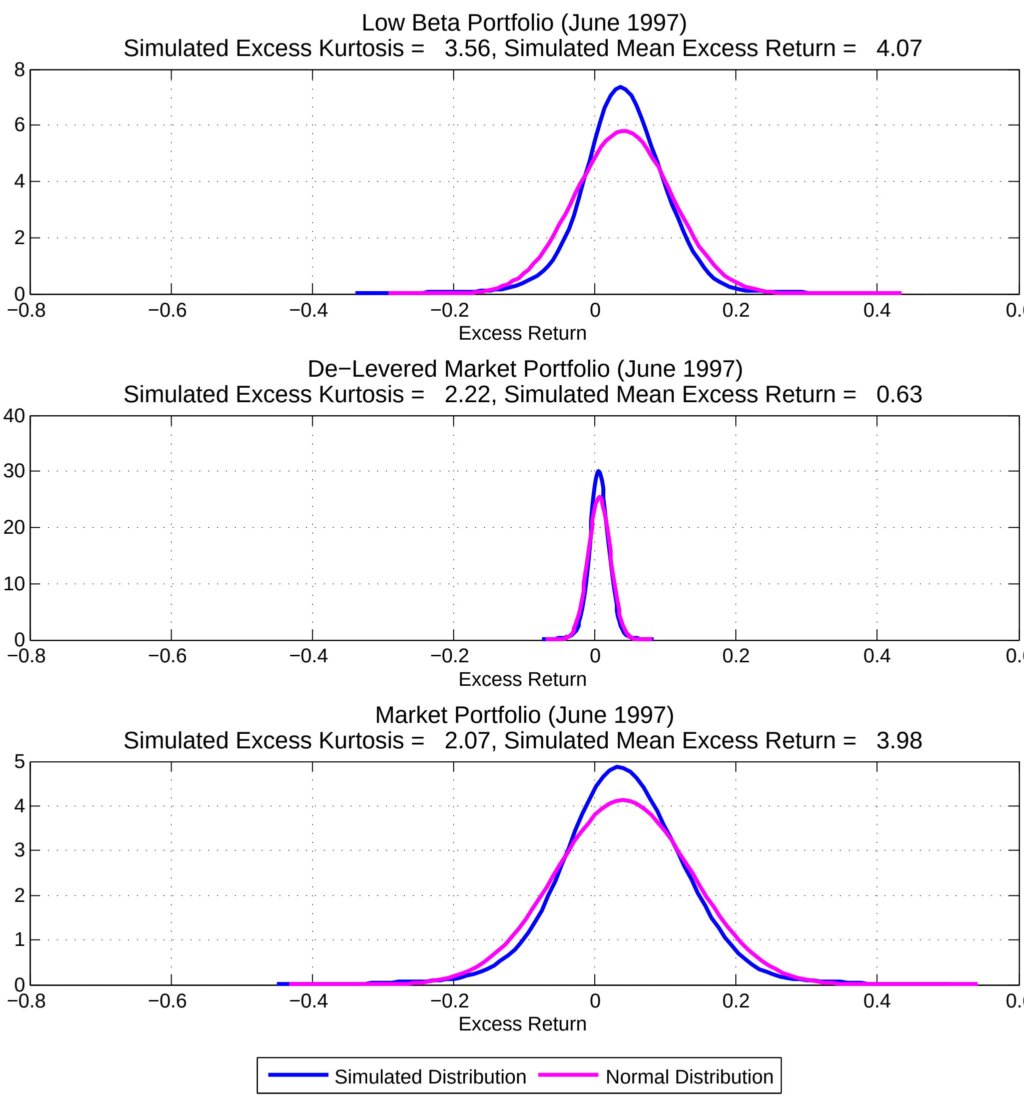


Figure 1.12: Simulated Kurtosis for Low-Beta, De-Levered Market, and Market Portfolios

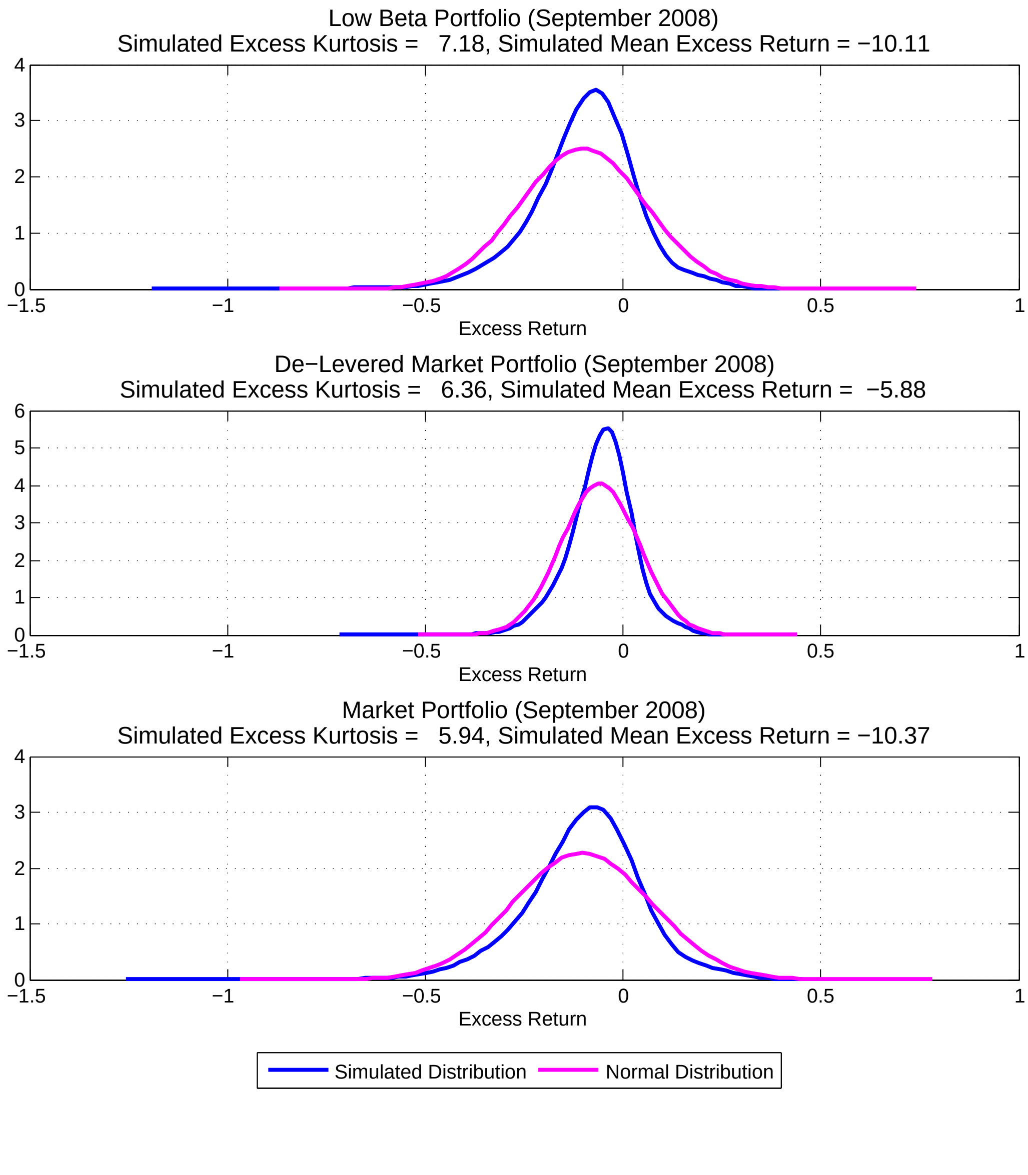


Figure 1.13: Estimates of λ_θ for $\theta \in \Theta$

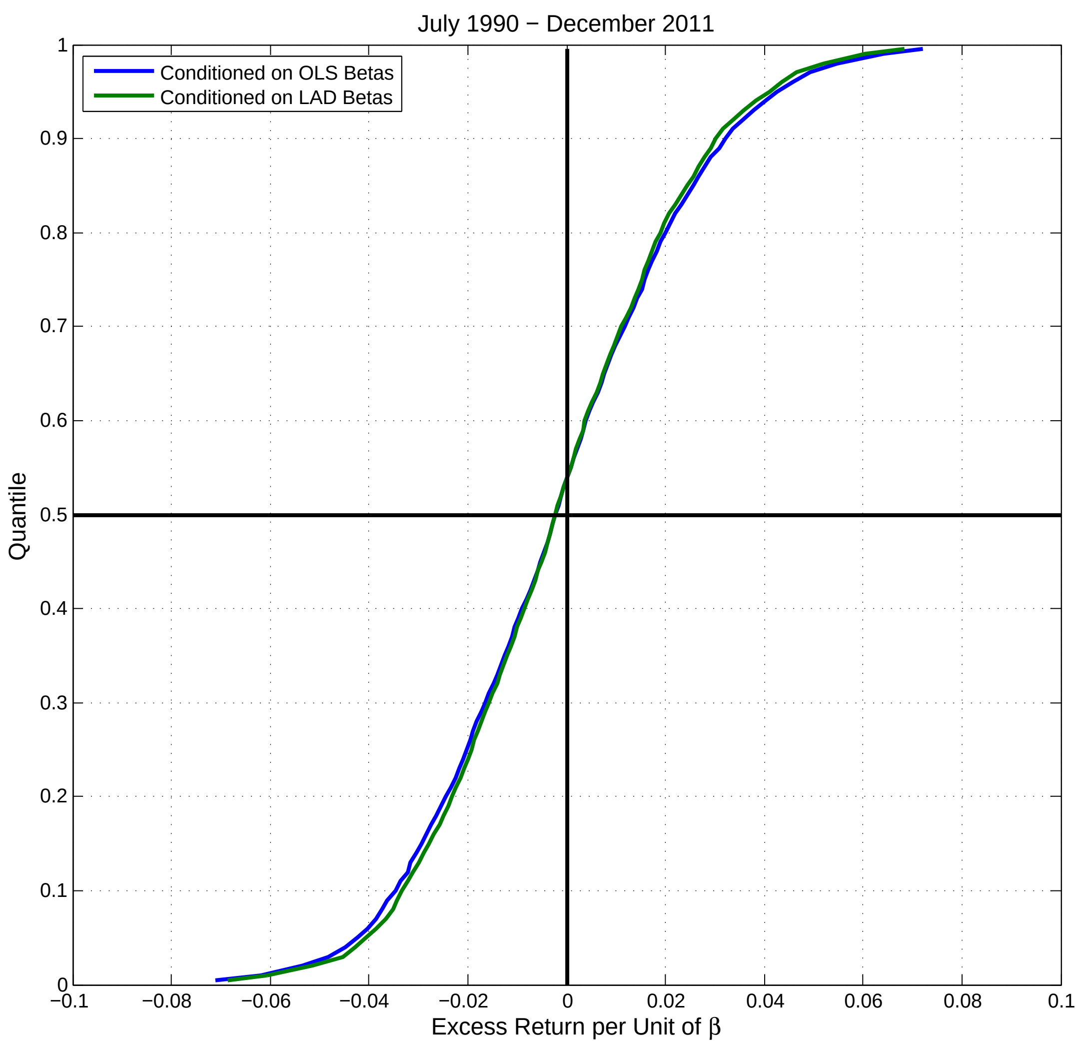
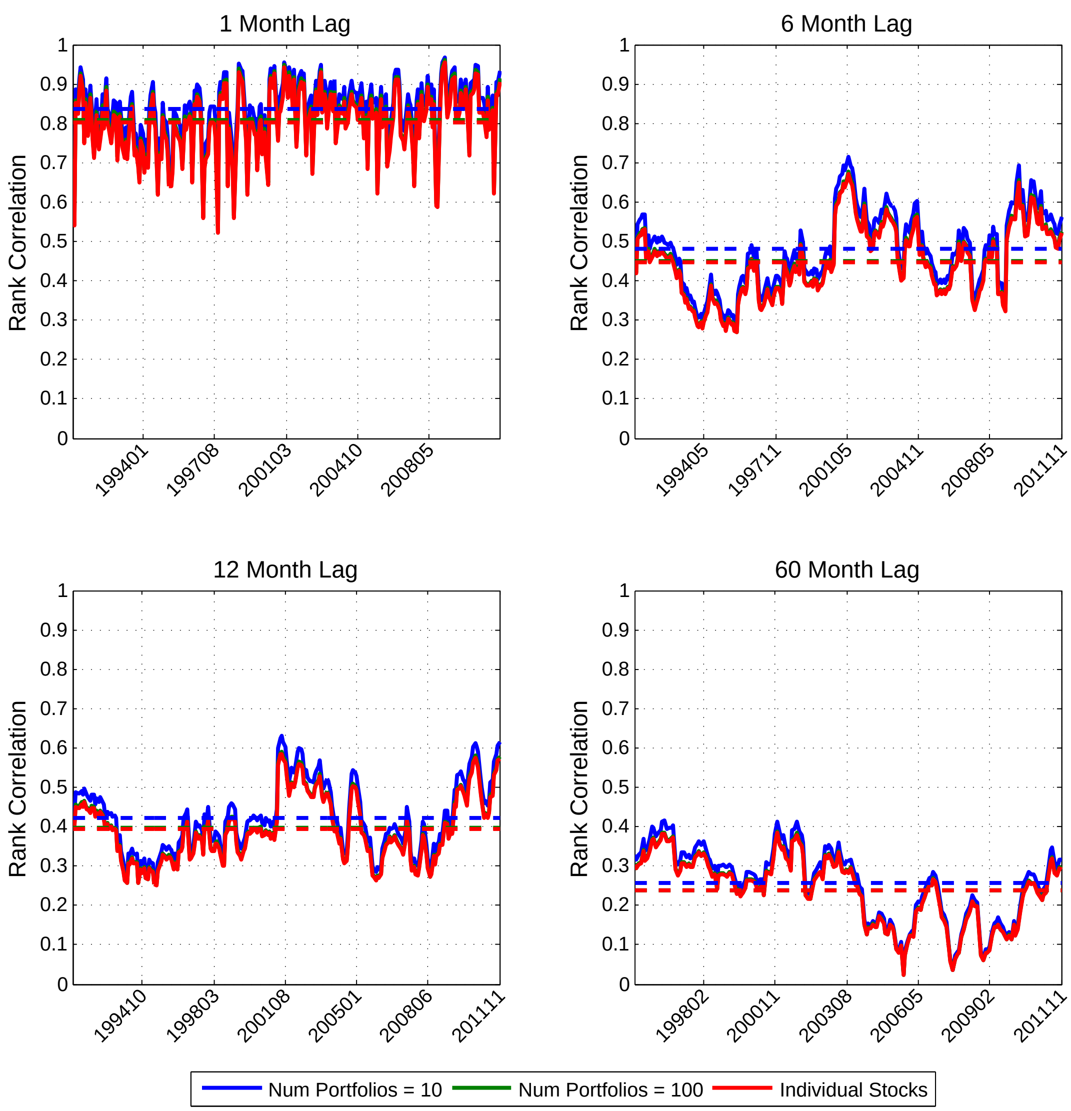


Figure 1.14: Beta Rank Correlations



Chapter 2

Levered Portfolios

In this chapter, we show that there are five elements that determine the cumulative return to a levered strategy, and they fit together in a simple, useful formula. Looking backward, our formula can be used to attribute performance. Looking forward, an investor can populate our formula with his or her forecast of the five determinants in order to generate a forecast for return to the levered strategy.

A levered strategy begins with a fully-invested source portfolio, such as unlevered risk parity, unlevered minimum variance, or unlevered bonds. The source portfolio is then levered according to a leverage rule. The most common leverage rules target volatility: they estimate the current volatility of the source portfolio in some way, and then choose leverage so that the estimated volatility of the levered strategy matches the target. Since the *source* portfolio typically exhibits variable volatility, volatility targeting requires *dynamic* leverage, even if the volatility target is *fixed*.

Much of our intuition about levered strategies comes from single-period models. In a single-period model, the return of the levered strategy is determined by the return of the source portfolio, the leverage, and the financing cost associated with the leverage. By definition, leverage is constant; there is no trade and hence no trading costs; and there is no compounding to take into account.

Now consider a simple two-period model. Assume that the source portfolio earns a 10% arithmetic return in period 1 and a -10% arithmetic return in period 2. We invest \$100.00 which is worth \$110.00 at the end of period 1 and \$99.00 at the end of period 2, as shown in table 2.1. The average of the arithmetic return over the two periods is zero, but the cumulative return of the source portfolio over the two periods is:

$$\frac{99-100}{100} = -0.01 = -1.00\%$$

The average arithmetic return of the source portfolio return must be corrected for compounding. As noted by Booth and Fama (1992) and discussed in Appendix B.4, the correction subtracts half the variance of arithmetic return each period; we call this corection the

Table 2.1: Strategies in the Two-Period Example

Time	Source Return r^S	Assets A (\$)	Debt D (\$)	Strategy Value $A - D$ (\$)
Source Portfolio:				
Beginning of 1		100.00	0.00	100.00
End of 1	10%	110.00	0.00	110.00
End of 2	-10%	99.00	0.00	99.00
Fixed Leverage Strategy:				
Beginning of 1		250.00	150.00	100.00
End of 1	10%	275.00	150.00	125.00
End of $1'$		312.50	187.50	125.00
End of 2	-10%	281.25	187.50	93.75
Dynamically Levered Strategy:				
Beginning of 1		200.00	100.00	100.00
End of 1	10%	220.00	100.00	120.00
End of $1'$		360.00	240.00	120.00
End of 2	-10%	324.00	240.00	84.00

Notes: Calculation of return of the source portfolio and levered strategies in the two-period model. The rows with time End of 1 represent the levered strategy prior to rebalancing, while the rows with time End of $1'$ represents the levered strategy after rebalancing.

variance drag. Note that the variance of the arithmetic returns is:

$$\frac{(0.1-0)^2 + (-0.1-0)^2}{2} = .01 = 1.00\%$$

If we subtract half the variance from the arithmetic return each period, we get a total return of:

$$0.1 - \frac{.01}{2} + (-0.1) - \frac{.01}{2} = -0.01 = -1.00\%$$

which matches the actual cumulative return over the two periods.

Now, consider a levered strategy. Suppose for simplicity that leverage can be financed at the risk-free rate, which happens to be zero. We initially invest \$100.00. Suppose that we target a fixed volatility of 12% per period, and our estimate of the source portfolio volatility is 6% at the beginning of the first period and 4% at the beginning of the second period. Thus, we choose leverage $\lambda_1 = \frac{12}{6} = 2$ in the first period and $\lambda_2 = \frac{12}{4} = 3$ in the second period.

If we try to squeeze this into a one-period analysis, we might be tempted to assume the return will be similar to that of a strategy with fixed leverage $\bar{\lambda} = \frac{\lambda_1+\lambda_2}{2} = 2.5$, since this is the average leverage over the two periods. Thus, we might expect to earn about 2.5 times the average arithmetic return of the source portfolio, or $2.5 \times 0.00\% = 0.00\%$. If we are a little more sophisticated and take compounding into account, we might expect to earn $2.5 \times -1.00\% = -2.50\%$. However, both these answers are wrong, even for the case of fixed leverage, and they are particularly wrong for the case of dynamic leverage.

Consider a fixed leverage strategy that uses leverage 2.5 in both periods. As noted in table 2.1, we hold assets of \$250.00 in the source portfolio (financed by our \$100.00 and \$150.00 in debt) at the beginning of the first period. At the end of the first period, our assets are worth \$275.00, and our debt is \$150.00, so the value of the levered strategy is \$125.00. Even though we want to maintain fixed leverage, we need to rebalance. We hold assets of $\$125.00 \times 2.5 = \$312.50$ in the source portfolio. We must borrow, increasing our debt to \$312.50-125.00=\$187.50 to finance the position. At the end of the second period, our assets in the source portfolio are worth $\$312.50 \times 0.9 = \$281.25$, so the value of the levered strategy is $\$281.25 - 187.50 = \$93.75$; our cumulative return over the two periods is $\frac{93.75-100}{100} = -0.625$, a loss of 6.25%. The variance of levered strategy return is: $\frac{(.25-0)^2+(-0.25-0)^2}{2} = .0625 = 6.25\%$ The correction for compounding (variance drag) is $\frac{6.25\%}{2}$ per period; over two periods, this gives -6.25%, exactly matching the realized return. Note that the variance drag is *quadratic* in leverage, so constant leverage of 2.5 increases the variance drag by a factor of 6.25.

With our more sophisticated understanding of the quadratic nature of the variance drag, we might expect the dynamically levered strategy to have a cumulative return of about -6.25%. However, that answer is also wrong. As shown in table 2.1, the dynamically levered strategy holds assets of \$200.00 in the source portfolio at the beginning of the first period, financed by our initial \$100.00 and debt of \$100.00. At the end of the first period, the assets are worth a total of \$220.00, the debt is still \$100.00, so the value of the levered strategy is \$120.00. We rebalance to achieve the prescribed leverage. Since the levered strategy calls for leverage $\lambda_2 = 3$, we borrow an additional \$140.00 for total debt of \$240.00, and hold assets of \$360.00 in the source portfolio. We incur trading costs, which for simplicity we assume to be zero. At the end of the second period, our shares of the source portfolio are worth $\$360.00 \times 0.9 = \$324.00$; since we owe \$240.00, so our equity is \$84.00. The cumulative return to the levered strategy over the two periods is:

$$\frac{84 - 100}{100} = -.16 = -16.00\%$$

Rather than breaking even, or losing 2.5%, or losing 6.25% as we expected from our single-period intuition, we have lost 16%.

We went wrong because we ignored the *covariance* between leverage λ and source portfolio return $r^{\mathbf{S}}$:

$$\begin{aligned}
\mathbf{C}ov(\lambda, r^{\mathbf{S}}) &= \frac{(\lambda_1 - \bar{\lambda}) \times (r_1^{\mathbf{S}} - 0) + (\lambda_2 - \bar{\lambda}) \times (r_2^{\mathbf{S}} - 0)}{2} \\
&= \frac{(2 - 2.5)(.1 - 0) + (3 - 2.5)(-0.1 - 0)}{2} \\
&= \frac{-0.05 - 0.05}{2} \\
&= -0.05 = -5.00\%
\end{aligned}$$

The covariance term reduces return by 5.00% *each period*, producing a return of -10.00%. The arithmetic return of the dynamically levered strategy is 0.2 in the first period and -0.3 in the second periods, so the variance of the return of the levered strategy is:

$$\frac{(.2 - (-0.05))^2 + (-0.3 - (-0.5))^2}{2} = \frac{(0.25)^2 + (-0.25)^2}{2} = 0.0625 = 6.25\%$$

The variance drag for the dynamically levered strategy is the same as for the fixed leverage strategy:[1] half the variance, or 3.125%, each period. Combining the covariance and variance, we get a return of -8.125%, per period, suggesting a cumulative loss of 16.25%, close to the cumulative actual return of -16.00% over the combination of the two periods.

As this example indicates, the covariance term can make a big difference over a few periods. One might be tempted to think the covariance term would wash out over time. If that were true, the covariance term might not be particularly important. Strikingly, we found that the covariance term made a *substantial difference* over a *very long horizon*. Our empirical examples include as source portfolios risk parity (with asset classes consisting of US stocks and US Treasury bonds) and Treasury bonds alone, with two different types of volatility targeting and two different volatility targets. In all of our examples, the covariance term turned out to be negative, subtracting between 0.64% and 4.23% from annualized return over an 84-year horizon. Consequently, the Sharpe ratios of volatility-targeting strategies were diminished relative to their source portfolios and fixed leverage benchmarks.

2.1 Synopsis of Theoretical Contributions and Empirical Findings

Hedge funds, real estate investment trusts, and many other investment vehicles routinely use leverage. Even among the most conservative and highly regulated investors such as US public pension funds, the use of levered investment strategies is widespread and growing.[2] In the

[1] This is true in this specific example, but is not true in general.

[2] See, for example, Kozlowski (2013).

period since the financial crisis, strategies such as risk parity that explicitly lever holdings of publicly traded securities have emerged as candidates for these investment portfolios.[3]

In the single-period Capital Asset Pricing Model (CAPM), the market portfolio is the unique portfolio of risky assets that maximizes the Sharpe ratio. Leverage serves only as a means to travel along the efficient frontier. Both excess return and volatility scale linearly with leverage, and a rational investor will lever or de-lever the market portfolio in accordance with his or her risk tolerance.

Empirically, certain low-volatility portfolios have exhibited higher Sharpe ratios than did the market portfolio,[4] which suggests that levering a low-volatility source portfolio could deliver an attractive risk-return tradeoff. However, market frictions such as the difference between borrowing and lending rates, and the correlations that arise in multi-period models make the relationship between the realized return of a levered strategy and the Sharpe ratio of its source portfolio both nuanced and complex. Levered strategies tend to have substantially higher transaction costs[5] than do traditional strategies.[6]

We develop an exact performance attribution for levered strategies that takes market frictions into account. Specifically, we show that there are five important elements to cumulative return. The first element is the return to the fully invested portfolio to be levered, which we call the *source portfolio*. The second element is the expected return to the source in excess of the borrowing rate, amplified by leverage minus one. We call the sum of these terms the *magnified source return*, and it represents the performance of a levered strategy in an idealized world.

In the real world, the magnified source return is enhanced or diminished by the covariance between leverage and excess borrowing return, which is the third element of cumulative return of a levered strategy. Empirically, the covariance term turned out to be unstable at medium horizons of three to five years. Looking back, this made certain levered strategies appear particularly appealing at some times and particularly disappointing at other times. Viewed prospectively, it added considerable noise to medium horizon returns. The fourth and fifth elements, the cost of trading and the variance drag, are familiar to many investors. We penalized trading according to a linear model and we estimated the variance drag, which is effectively the difference between arithmetic and geometric return, using a formula that is adapted from Booth and Fama (1992).

Section 2.2 provides the foundation for our performance attribution, which is derived in section 2.2.1. In section 2.2.2, we illustrate the performance attribution in the context of

[3]Sullivan (2010) discusses the risks that a pension fund incurs by employing a levered strategy.

[4]See chapters 1 and 3 for examples.

[5]Investment returns are often reported gross of fees and transaction costs. That practice may be reasonable in comparing strategies with roughly equal fees or transaction costs, but it is inappropriate when comparing strategies with materially different fees or transaction costs.

[6]By traditional strategies, we mean the strategies that have typically been employed over the last 50 years by pension funds and endowments. These strategies invest, without leverage, in a relatively fixed allocation among asset classes.

a particular risk parity strategy, $\mathrm{UVT}_{60/40}$, which targeted a fixed volatility equal to the realized volatility of a 60/40 fixed mix over our 84-year sample period, 1929–2012. As shown in table 2.2, all five terms in the performance attribution contributed materially to the cumulative return of $\mathrm{UVT}_{60/40}$. For example, the covariance term subtracted an average of 1.84% per year from the expected arithmetic return of the magnified source portfolio.

Section 2.3 discusses the assumptions we made about historical borrowing and trading costs and their impact on performance comparisons.

Our performance attribution facilitates a comparison between a levered strategy and a variety of benchmarks, which are explored in section 2.4. The benchmarks fall into two classes. The first consists of fully invested portfolios, while the second consists of portfolios that use fixed leverage. For example, we compare $\mathrm{UVT}_{60/40}$ to its source portfolio and its volatility target, a 60/40 fixed mix. The comparison of a levered strategy to fully invested benchmarks is important since there would be no rational reason to invest in a levered strategy if it underperformed these benchmarks. However the comparison is clouded by the fact that backtests of levered strategies rely on assumptions about historical financing costs, while backtests of unlevered strategies do not.[7]

By contrast, comparisons among backtests of different types of levered strategies are on firmer ground: even if there are errors in the assumptions about financing costs, they affect all the strategies under consideration in similar ways. We introduce two fixed-leverage strategies in section 2.4.2. The first, $\mathrm{FLT}_{60/40,\lambda}$ had constant leverage equal to the average leverage of $\mathrm{UVT}_{60/40}$; the second, $\mathrm{FLT}_{60/40,\sigma}$, had constant leverage and had volatility equal to the volatility of $\mathrm{UVT}_{60/40}$. In our backtests, the fixed leverage strategies outperformed $\mathrm{UVT}_{60/40}$ as well as a conditional volatility-targeting risk parity strategy, $\mathrm{CVT}_{60/40}$, which is also introduced in section 2.4.2. The volatility-targeting strategies had lower Sharpe ratios than the corresponding fixed-leverage strategies, which had lower Sharpe ratios than the underlying source portfolios. Section 2.4.3 discusses how the levered strategies UVT, CVT, and FLT responded to changes in market conditions; in particular, with $\lambda > 1$, these turned out to be momentum strategies.

$\mathrm{CVT}_{60/40}$ matched the contemporaneous volatility of the fixed-mix 60/40, rather than its unconditional volatility over a long horizon. An advantage of $\mathrm{CVT}_{60/40}$ over the other strategies is that is investable: perfect foresight is not required to rebalance the strategy each month. On the other hand, UVT and FLT strategies can be set by choosing a fixed volatility or leverage that is in the ball park of the expected future volatility of the target. This raises the question of sensitivity to parameters: if we set a UVT target volatility with an *intent* to match the volatility of a given strategy, such as the value-weighted market or 60/40, how close will the performance of the strategy we implement be to the performance of the strategy we intended to implement? We do not seriously address this question here,

[7]To the extent that levered strategies exhibited higher turnover than fully invested strategies, their returns may have been more sensitive to assumptions about historical trading costs. In our empirical results, financing costs had a significantly greater impact than did trading costs.

but a hint about its complexity and depth is in section 2.4.4, which looks at the impact of the target volatility on strategy performance. These four risk parity strategies lever a common source portfolio, so it is straightforward to compare the return attributions of the strategies. The details are in table 2.6, which shows, for example, that the covariance drag in $\mathrm{UVT}_{60/40}$ was substantially larger than in $\mathrm{CVT}_{60/40}$, and the difference in the covariance drags of $\mathrm{UVT}_{\mathrm{MKT}}$ and $\mathrm{CVT}_{\mathrm{MKT}}$ was even more pronounced. The high magnitude of the covariance drag and its sensitivity to the volatility target in UVT, came from both a high volatility of leverage and high sensitivity of the volatility of leverage to the UVT volatility target, compared to CVT.

In section 2.4.5, we look beyond risk parity by considering a US government bond index levered to the volatility of US equities. The results are qualitatively similar although they are more dramatic since the volatility of the source portfolio is lower in this example than in the others, while the target volatility is higher. The results are in table 2.7. For example, the covariance term in $\mathrm{UVTB}_{\mathrm{STOCKS}}$ subtracted 4.23% per year from strategy performance.

Section 2.5 revisits the covariance term from the viewpoint of volatility-targeting. It demonstrates that the covariance term is still present from the volatility-matching perspective, and demonstrates that fixed-volatility targeting is a form of unintended market-timing, whereas fixed leverage is not.

In all of the volatility-targeting strategies we considered, the covariance term in the UVT strategies was negative over our 84-year data period. We note that the UVT covariance term was positive over many three-to-five year periods, and some periods lasting two to four decades.

Section 2.6 summarizes our main conclusions.

We also include a number of appendices that support our main narrative. Appendix B.1 provides a detailed overview of the literature on low-risk investing and leverage. Appendix B.2 describes the data in enough detail to allow researchers to replicate our results. Appendix B.3 describes our linear trading model. Appendix B.4 derives our approximation of geometric return from arithmetic return. As illustrated in our empirical examples, this approximation has a high degree of accuracy in practical situations. Appendix B.5 presents a table with the formulas and corresponding words for the elements of our performance attribution.

2.2 The Impact of Leverage on the Return to an Investment Strategy

Leverage magnifies return, but that is only one facet of the impact that leverage has on an investment strategy. Leverage requires financing and exacerbates turnover, thereby incurring transaction costs. It amplifies the variance drag on cumulative return due to compounding. When leverage is dynamic, it can add substantial noise to strategy return. We provide an exact attribution of the cumulative return to a levered strategy that quantifies these effects.

A *levered strategy* is built from a fully invested *source portfolio* of risky assets, presumably chosen for its desirable risk-adjusted returns, and a *leverage rule*.[8]

An investor has a certain amount of capital, L. The investor chooses a leverage ratio λ, borrows $(\lambda - 1)L$, and invests λL in the source portfolio.[9]

In what follows, we assume $\lambda > 1$.

2.2.1 Attribution of Arithmetic and Geometric Return

The relationship between the single-period return to a levered portfolio, $r^{\mathbf{L}}$, and to its source portfolio, $r^{\mathbf{S}}$, is given by:

$$r^{\mathbf{L}} = \lambda r^{\mathbf{S}} - (\lambda - 1) r^{b}, \tag{2.1}$$

where the borrowing rate, r^b, is greater than or equal to the risk-free rate r^f. Note that the excess return is given by:

$$\begin{aligned} r^{\mathbf{L}} - r^{f} &= \lambda r^{\mathbf{S}} - (\lambda - 1) r^{b} - r^{f} \\ &= \lambda \left(r^{\mathbf{S}} - r^{f} \right) - (\lambda - 1) \left(r^{b} - r^{f} \right) \end{aligned} \tag{2.2}$$

Excess return and volatility scale linearly in λ for $\lambda \geq 0$ if and only if $r^b = r^f$; in that case, the situation is essentially the same as the single-period CAPM, except that the source portfolio need not be the market portfolio.

When $r^b > r^f$, volatility still scales linearly in $\lambda \geq 0$ but formula (2.2) indicates that excess return scales sublinearly; as a consequence, the Sharpe ratio is a *declining* function of λ. Note that the excess borrowing return of the levered strategy is:

$$r^{\mathbf{L}} - r^{b} = \lambda \left(r^{\mathbf{S}} - r^{b} \right) \tag{2.3}$$

It is the excess *borrowing* return and volatility that scale linearly in leverage, for $\lambda \geq 1$. The bar for leverage to have a positive impact on return has gotten higher: the excess borrowing return, $r^{\mathbf{S}} - r^{b}$, must be positive.

The expected return to a levered strategy is estimated by rewriting formula (2.3) as:

$$r^{\mathbf{L}} = r^{\mathbf{S}} + (\lambda - 1) \left(r^{\mathbf{S}} - r^{b} \right) \tag{2.4}$$

[8] The source portfolio can be long-short in the risky assets. It must, however, have a non-zero value, so that returns can be calculated. Since we want to model leverage explicitly, we do not allow the source portfolio to contain a long or short position in a riskless asset, such as T-bills, the money market account, or commercial paper.

[9] Leverage may be achieved through explicit borrowing. It may also be achieved through the use of derivative contracts, such as futures. In these derivative contracts, the borrowing cost is implicit rather than explicit, but it is real and is typically at a rate higher than the T-Bill rate. For example, Naranjo (2009) finds that the implicit borrowing cost using futures is approximately the applicable LIBOR rate, applied to the notional value of the futures contract.

and taking the expectation over multiple periods:

$$\begin{aligned} E\left[r^{\mathbf{L}}\right] &= E\left[r^{\mathbf{S}}\right] + E\left[(\lambda-1)\left(r^{\mathbf{S}}-r^{b}\right)\right] \\ &= E\left[r^{\mathbf{S}}\right] + E\left[\lambda-1\right]E\left[r^{\mathbf{S}}-r^{b}\right] + \mathbf{C}ov(\lambda, r^{\mathbf{S}}-r^{b}) \end{aligned} \tag{2.5}$$

We use the term *magnified source return* to denote the sum of the first two terms on the right side of formula (2.5). That formula shows that the expected return to a levered strategy is equal to the magnified source return plus a covariance correction. We find empirically that, even when the correlation between leverage and excess borrowing return is quite small, the covariance correction can be substantial in relation to the magnified source return.

We can interpret the expectation and covariance in formula (2.5) in two ways. Prospectively, they represent the expectation and covariance under the true probability distribution. Retrospectively, they represent the realized mean and realized covariance of the returns.[10]

Also important over multiple periods is the cost of trading, which imposes a drag $r^{\mathbf{TC}}$ on any strategy: To take account of this effect, we extend formula (2.5):

$$\begin{aligned} E\left[r^{\mathbf{L}}\right] &= E\left[r^{\mathbf{S}}\right] + E\left[\lambda-1\right]E\left[r^{\mathbf{S}}-r^{b}\right] + \mathbf{C}ov(\lambda, r^{\mathbf{S}}-r^{b}) - E\left[r^{\mathbf{TC}}\right] \\ &= E\left[r^{\mathbf{S}}\right] + E\left[\lambda-1\right]E\left[r^{\mathbf{S}}-r^{b}\right] + \mathbf{C}ov(\lambda, r^{\mathbf{S}}-r^{b}) - \left(E\left[r^{\mathbf{TCS}}\right] + E\left[r^{\mathbf{TCL}}\right]\right) \end{aligned} \tag{2.6}$$

where $r^{\mathbf{TC}}$ is expressed as a sum of trading costs due to turnover in the source portfolio and trading costs due to leverage-induced turnover:

$$r^{\mathbf{TC}} = r^{\mathbf{TCS}} + r^{\mathbf{TCL}}.$$

Estimates of $r^{\mathbf{TC}}$ and its components rely on assumptions about the relationship between turnover and trading cost. We assumed that cost depended linearly on the dollar value that turned over, and we used formulas (B.5) and (B.6) to estimate $r^{\mathbf{TC}}$ in our empirical studies. More information is in Appendix B.3.

Formula (2.6) is based on arithmetic expected return, which does not correctly account for compounding. The correction for compounding imposes a variance drag on cumulative return that affects strategies differentially; for any given source portfolio, the variance drag is quadratic in leverage. If a levered strategy has high volatility, the variance drag may be substantial.

If we have monthly returns for months $t = 0, 1 \ldots, T-1$ the realized geometric average of the monthly returns is:

$$G[r] = \left(\prod_{t=0}^{T-1}(1+r_t)\right)^{1/T} - 1 \tag{2.7}$$

[10]Note that we take the realized covariance, obtained by dividing by the number of dates, rather than the realized sample covariance, which would be obtained by dividing by one less than the number of dates. We use the realized covariance because it makes formula (2.5) true.

where r_t is the arithmetic return in month t. Given two strategies, the one with the higher realized geometric average will have higher realized cumulative return. In Appendix B.4, we show that the following holds to a high degree of approximation:[11]

$$G[r] \quad \sim \quad (1 + E[r])\, e^{-\frac{\mathbf{V}ar(r)}{2}} - 1 \tag{2.8}$$

Note that the correction depends only on the realized variance of return.[12] Booth and Fama (1992) provide a correction for compounding based on continuously compounded return; our correction for the geometric average of monthly returns in formula (2.8) is slightly simpler.

Thus, in comparing the realized returns of strategies, the magnified source return of the levered strategy must be adjusted for three factors that arise only in the multi-period setting: the covariance correction, the variance drag, and trading costs.[13]

2.2.2 Empirical Example: Performance Attribution of a Levered Risk Parity Strategy

We demonstrate the utility of the performance attribution detailed above in the context of $\mathrm{UVT}_{60/40}$, a risk parity strategy that was rebalanced monthly and levered to an unconditional volatility target equal to the realized volatility, 11.59%, of the 60/40 fixed-mix between January 1929 and December 2012.[14] The source portfolio was unlevered risk parity based on two asset classes, US Equity and US Treasury Bonds. Foresight was required in order to set this target: the volatility of the 60/40 strategy was not known until the end of the period.[15]

Figure 2.1 shows the magnified source return and the realized cumulative return to $\mathrm{UVT}_{60/40}$, as well as the realized cumulative return to its source portfolio (fully invested risk parity) and target (60/40 fixed mix). All computations assumed that leverage is financed at the 3-month Eurodollar deposit rate. The realized cumulative returns were based

[11] The magnitude of the error is estimated following formula (B.10). Note that G and E denote realizations of the geometric and average arithmetic return, respectively. The term $\mathbf{V}ar(r)$ denotes the realized variance of r, rather than the realized sample variance.

[12] In an earlier version of this paper, we indicated, incorrectly, that both the level and the variability of volatility determine the magnitude of the variance drag.

[13] Note that the source and target portfolios may incur their own trading costs, as well as benefit from volatility pumping. The performance attribution of formula (2.6) uses the source return and magnified source return, *gross* of trading costs. When we report historical arithmetic returns to the source and target portfolio, we report these net of trading costs, and inclusive of any benefit from volatility pumping. When we report cumulative returns to the source and target portfolios, we report these net of the variance drag.

[14] The leverage was chosen so that the volatility, gross of trading costs, was exactly 11.59%. When trading costs were taken into account, the realized volatility was slightly lower: 11.54%. $\mathrm{UVT}_{60/40}$ was constructed in effectively the same way as the levered risk parity strategy in Asness et al. (2012), with one main difference. They levered risk parity to match the volatility of the market, which had higher volatility than 60/40. In section 2.4.4, we consider risk parity levered to the volatility of the market.

[15] The sensitivity of strategy performance to the volatility target is discussed in section 2.4.4.

on the additional assumption that trading is penalized according to the linear model described in Appendix B.3, and took into account the covariance correction and variance drag on cumulative return. The magnified source return of $\mathrm{UVT}_{60/40}$ easily beat the cumulative return of both the source and the target; however, the realized cumulative return of $\mathrm{UVT}_{60/40}$ was well below the realized cumulative return of the 60/40 target portfolio (with essentially equal volatility (11.58%)) and only slightly better than *unlevered* risk parity source portfolio, which had much lower volatility (4.20%).[16]

The return decomposition formulas (2.6) and (2.8) provide a framework for analyzing the performance of $\mathrm{UVT}_{60/40}$. Table 2.2 provides the required information. Consider first the magnified source return. The source portfolio had an annualized arithmetic return of 5.75% gross of trading costs.[17] Leverage added an extra 3.97% to annualized return from the magnification term, the average excess borrowing return to the source portfolio multiplied by average leverage minus one. The annualized magnified source return was thus 9.72%. However, the covariance between leverage and excess borrowing return reduced the annualized return by 1.84%, trading costs by 96 basis points, and variance drag by a further 48 basis points. Together, these three effects ate up 3.28% of the 3.97%, or 82.6%, of the contribution of leverage to the magnified source return.

2.3 Assumptions about Transaction Costs and Their Impact on Empirical Results

The return calculations in our empirical examples relied on assumptions about transaction costs over our study period, 1929–2012. Comparisons between levered and unlevered strategies were sensitive to these assumptions, but comparisons between strategies that were comparably levered were much less sensitive to them. For transparency, we include the details of our assumptions about transaction costs in Appendices B.2 and B.3. Here, we explain some of the reasoning that led to the choices we made, and we discuss the impact of our choices on the results.

One guideline is that trading became less expensive over time during the study period, so we assessed a greater cost to turnover at the beginning of the period than the end. Specifically, we assumed that the portfolio was rebalanced monthly[18] and that trading cost 1% of the dollar amount of a trade between 1929 and 1955, .5% between 1956 and 1971, and .1% between 1972 and 2012. Since turnover tended to be higher in a levered strategy than in an unlevered strategy, higher trading costs tended to do more damage to a levered strategy

[16] The volatilities are reported in table 2.3.

[17] Trading costs subtracted only 7 basis points per year from the source return.

[18] In practice, trading costs can be reduced by reducing the frequency or completeness of rebalancing, at the cost of introducing tracking error. Further, trading costs may be higher for some asset classes than for others. However, in our empirical examples, financing costs were more important than trading costs.

than to an unlevered strategy.

As a borrowing rate, we used the 3-month Eurodollar deposit rate, for which we had data back to the beginning of 1971. Prior to 1971, we used the 3-month T-bill rate plus a spread of 60 basis points, which was 40 basis points less than the average spread between the Eurodollar deposit rate and the T-bill rate between 1971 and 2012. This choice improved the performance of our levered strategies relative to what they would have been had we used the average spread. Of course, a lower borrowing rate would have further improved the performance of the levered strategies.[19] Since the levered strategies involved borrowing and the unlevered strategies did not, there were more assumptions underlying the empirical results for levered strategies than for unlevered strategies. As a consequence, our uncertainty about results for levered strategies was greater than for unlevered strategies.

It would, of course, have been possible to include empirical results based on a wider range of assumptions about transaction costs. However, that would have been misleading since it would have conveyed the impression that we had done a thorough study of the issue. We did not. We chose a streamlined approach of providing examples based on single set of assumptions that are consistent with published literature and that rely on readily available data. The purpose of these examples is to illustrate the efficacy of our performance attribution framework. We encourage practitioners and scholars to apply our framework using their own estimates of trading and borrowing costs in order to evaluate strategies and to facilitate the decision to lever.

2.4 Benchmarks for a Levered Strategy

2.4.1 Fully Invested Benchmarks

Table 2.3 reports annualized arithmetic and geometric return, volatility and Sharpe ratio to $\mathrm{UVT}_{60/40}$, its source, and its target. Because $\mathrm{UVT}_{60/40}$ was levered, while the source and target were not, these comparisons were subject to uncertainty about historic financing and trading costs. $\mathrm{UVT}_{60/40}$ had annualized geometric return only 63 basis points higher than the source portfolio, unlevered risk parity.[20] At the same time, the source portfolio had a

[19] We considered using 1-month rates, but that would have engendered a more complex extrapolation since the 1-month T-bill rate began only in 2001. Note that the difference between the 1-month and 3-month Eurodollar deposit rates averaged 20 basis points between 1971 and 2013. This was offset by the 40 basis points we subtracted in our extrapolation.

[20] Note that the annualized geometric return of the source portfolio, 5.74%, slightly exceeded 5.68%, the annualized arithmetic return of the source portfolio, net of trading costs. This is an artifact of the annualization procedures for arithmetic and geometric return. The source portfolio had monthly arithmetic return of 47.3 basis points, net of transaction costs. The latter was annualized by multiplying by 12: $12 \times 0.473\% = 5.68\%$. Annualized geometric return takes into account compounding: $1.00473^{12} - 1 = 5.83\%$. The variance drag reduced this by 9 basis points to 5.74%. The variance drag on the source return was much smaller than the variance drag on the levered portfolios, because the source portfolio was so much less volatile and the

much lower volatility (4.20%). As a result, $\text{UVT}_{60/40}$ had a Sharpe ratio of 0.29, compared to 0.52 for unlevered risk parity. Note that the high Sharpe ratio of unlevered risk parity was obtained at the cost of low expected return.

60/40 and $\text{UVT}_{60/40}$ had essentially equal volatilities. Under our assumptions on historic financing and trading costs, 60/40 delivered an annualized geometric return of 7.77% and a realized Sharpe ratio of 0.40, while the analogous figures for $\text{UVT}_{60/40}$ were 6.37% and 0.29. Investors who are considering an investment in risk parity or any levered strategy can populate tables 2.2 and 2.3 with their forward-looking estimates of the components of strategy return. This analysis can inform the decision to invest in a levered strategy instead of the fully invested source or target portfolio.

2.4.2 Fixed Leverage and Conditional Leverage Benchmarks

In this section, we focus on comparisons of realized returns among levered strategies that were constructed in different ways. These comparisons were less sensitive to the assumptions on historical financing and trading costs. Like any volatility targeting strategy, $\text{UVT}_{60/40}$ was dynamically levered. However, as we saw in section 2.2.2, the covariance between leverage and excess borrowing return diminished annualized arithmetic return by 1.84%. Deeper insight into this cost is provided in table 2.2, which decomposes these covariances into products of correlation and standard deviations. Note that the magnitude of the correlation between leverage and excess borrowing return was small: -0.056. Figure 2.2 shows rolling 36-month estimates of the correlation between leverage and excess borrowing return, and indicates that the sign of the correlation flipped repeatedly at short horizons. At investment horizons of three to five years, the main effect of the covariance term appeared to be to add noise to the returns.

When leverage is fixed, the covariance between leverage and excess borrowing return must be zero. We consider two fixed leverage strategies: $\text{FLT}_{60/40,\lambda}$ matched the average leverage of $\text{UVT}_{60/40}$, but had higher volatility, while $\text{FLT}_{60/40,\sigma}$ matched the volatility of $\text{UVT}_{60/40}$ but had lower leverage.

Another alternative to UVT is a conditional volatility targeting strategy. $\text{CVT}_{60/40}$ levered fully invested risk parity so that the projected volatility (based on the previous 36 months returns) equalled the volatility of the target 60/40 over the previous 36 months.[21]

Table 2.4 provides performance attributions for $\text{UVT}_{60/40}$, $\text{FLT}_{60/40,\lambda}$, $\text{FLT}_{60/40,\sigma}$ and $\text{CVT}_{60/40}$. Note that each column of table 2.4 is a version of table 2.2 applied to one of our four levered strategies. All four levered strategies made use of the same source portfolio, and hence had the same source arithmetic return. Leverage contributed substantially and at roughly the same level to the magnified source return of $\text{UVT}_{60/40}$, $\text{FLT}_{60/40,\lambda}$ and $\text{CVT}_{60/40}$, since those three strategies had similar average leverage. The contribution to the return

variance drag is quadratic in volatility.

[21] $\text{CVT}_{60/40}$ is detailed in chapter 3.

of $\mathrm{FLT}_{60/40,\sigma}$ was significantly lower because that strategy had lower average leverage. The covariance term reduced the annualized arithmetic return of $\mathrm{UVT}_{60/40}$ by 1.84%, but led to a much smaller reduction in the return of $\mathrm{CVT}_{60/40}$ and, by design, had no effect on the return of the two FLT strategies. Trading costs reduced the return of $\mathrm{UVT}_{60/40}$ and $\mathrm{CVT}_{60/40}$ by about 95 basis points, but had a smaller effect on the two FLT strategies.[22] The variance drag reduced the geometric returns of $\mathrm{UVT}_{60/40}$, $\mathrm{FLT}_{60/40,\sigma}$ and $\mathrm{CVT}_{60/40}$ by similar amounts, since these strategies had similar variances; the effect on $\mathrm{FLT}_{60/40,\lambda}$ was greater as a result of its higher volatility. When all the effects were taken into account, the geometric returns of $\mathrm{FLT}_{60/40,\lambda}$, $\mathrm{FLT}_{60/40,\sigma}$ and $\mathrm{CVT}_{60/40}$ exceeded the geometric return of $\mathrm{UVT}_{60/40}$ by 192, 125 and 66 basis points, respectively.

2.4.3 Attributes of Levered Strategies

The parameters of the UVT and two FLT levered strategies were set with foresight. The dynamically levered strategy $\mathrm{UVT}_{60/40}$ was based on the realized volatility of a 60/40 fixed mix between January 1929 and December 2012. That volatility was known only at period end even though it was used to make leverage decisions throughout the period. The $\mathrm{FLT}_{60/40,\lambda}$ leverage was set to match the average leverage of $\mathrm{UVT}_{60/40}$ and the $\mathrm{FLT}_{60/40,\sigma}$ leverage was set so that the volatility matched the volatility of $\mathrm{UVT}_{60/40}$.

$\mathrm{CVT}_{60/40}$, introduced in section 2.4.2, did not rely on future information to set leverage.[23] As a result, its realized volatility failed to match the realized volatility of the target. At each monthly rebalancing, $\mathrm{CVT}_{60/40}$ was levered to match the volatility of the 60/40 fixed mix; both volatilities were estimated using a 36-month rolling window.

All else equal, $\mathrm{UVT}_{60/40}$, $\mathrm{FLT}_{60/40,\lambda}$, $\mathrm{FLT}_{60/40,\sigma}$ and $\mathrm{CVT}_{60/40}$ called for additional investment in the source portfolio when its price rose. A decline in the value of the source portfolio reduced the net value of the levered portfolio, while keeping the amount borrowed constant; leverage had increased, and rebalancing required selling the source portfolio to return to leverage λ. Similarly, an increase in the value of the source portfolio resulted in taking on more debt and using the proceeds to buy more of the source portfolio. In this sense, the UVT, FLT and CVT strategies with $\lambda > 1$ were momentum strategies. UVT, FLT and CVT strategies responded differently to changes in asset volatility; see table 2.5.

[22] As discussed in section 2.4.3 below, even maintaining a fixed leverage requires trading. It is possible in principle that the trading needed to adjust leverage to meet a volatility target could offset some of the trading required to maintain fixed leverage, but this strikes us as unlikely in typical situations. Had we assumed lower trading costs, it would have narrowed the gap in trading costs among the strategies, but not changed the ranking of those costs.

[23] The foresight in the definitions of UVT and the two FLT strategies allowed them to exactly match their volatility or leverage targets, gross of trading costs. Since $\mathrm{CVT}_{60/40}$ did not rely on foresight, it could not exactly match the realized target volatility, gross of trading costs. Both UVT and $\mathrm{CVT}_{60/40}$ volatility and FLT leverage were further affected by trading costs.

2.4.4 Changing the Volatility Target

In this section, we explore the relationship between UVT and CVT strategies, and in particular their sensitivity to the volatility target. In addition to 60/40, we used the Market Portfolio (i.e. the value-weighted portfolio of stocks and bonds, which has a higher volatility than 60/40) as the volatility target. UVT_{MKT} and CVT_{MKT} denote unconditionally levered and conditionally levered risk parity strategies with the market as the volatility target. Return comparisons of UVT_{MKT} to CVT_{MKT} and of $\text{UVT}_{60/40}$ to $\text{CVT}_{60/40}$ were not sensitive to our assumptions on historical financing and trading costs, while the comparisons of UVT_{MKT} to $\text{UVT}_{60/40}$ and of CVT_{MKT} to $\text{CVT}_{60/40}$ were only slightly sensitive to those assumptions.

Each term in the return attribution of the UVT risk parity strategies was sensitive to the choice of MKT or 60/40 as the volatility target. By contrast, the magnified source returns, covariance terms and trading costs of CVT_{MKT} were quite similar to those of $\text{CVT}_{60/40}$; the only large difference between the two CVT strategies lay in the variance drag. This finding indicates that CVT strategies were more stable than UVT strategies.

The geometric returns of UVT_{MKT} (6.53%) and CVT_{MKT} (6.52%) were virtually tied, while $\text{CVT}_{60/40}$ outperformed $\text{UVT}_{60/40}$ by 63 basis points.[24]

2.4.5 Changing the Source Portfolio

Thus far, we have illustrated our performance attribution on a variety of risk parity strategies that share a common source portfolio, unlevered risk parity. That allowed us to isolate the impact of different leverage rules on performance.

In this section, we examine the impact of the source portfolio on performance: we consider strategies that levered an index of US government bonds to target the volatility of US equities. As in the previous examples, we consider both a dynamically levered volatility targeting strategy, $\text{UVTB}_{\text{STOCKS}}$, as well as fixed leverage benchmarks, $\text{FLTB}_{\text{STOCKS},\lambda}$ (with the same average leverage as $\text{UVTB}_{\text{STOCKS}}$) and $\text{FLTB}_{\text{STOCKS},\sigma}$ (with the same volatility as $\text{UVTB}_{\text{STOCKS}}$). The details, presented in table 2.7, were qualitatively similar to what we saw for the risk parity strategies in tables 2.4 and 2.6: an attractive magnified source return was diminished substantially by transaction costs for all levered strategies and by the covariance term for the dynamically levered strategy, $\text{UVTB}_{\text{STOCKS}}$. However, since the source portfolio had lower volatility than unlevered risk parity, and the target volatility was higher than that of 60/40 and the value-weighted market, leverage was higher and the effects were more dramatic.

The covariance term for $\text{UVTB}_{\text{STOCKS}}$ was -4.23% per year, which imposed a larger drag on return than did the covariance terms (-1.84% and -2.73%) for $\text{UVT}_{60,40}$ and UVT_{MKT}. Despite the fact that the volatility target in $\text{UVTB}_{\text{STOCKS}}$ was *fixed*, the leverage was highly

[24] These findings do not support the assertion by Asness et al. (2013) that CVT is an inherently inferior implementation of risk parity, compared to UVT.

variable due to changes in the inverse of the volatility of the source portfolio of U.S. Treasury bonds.[25]

The correlation between leverage and excess borrowing return to the source portfolio was -.07. So as in the case of the dynamically levered risk parity strategies, a small correlation resulted in a large return drag. The geometric returns to $\mathrm{FLTB}_{\mathrm{STOCKS},\lambda}$ and $\mathrm{FLTB}_{\mathrm{STOCKS},\sigma}$ over our 84-year horizon were 5.93% and 6.94% per year. The geometric return to $\mathrm{UVTB}_{\mathrm{STOCKS}}$ over the same period was 1.7% per year.

2.4.6 Historical Performance of the Various Levered and Fully Invested Strategies

Table 2.8 summarizes the historical performance of our source portfolios (unlevered risk parity and U.S. Treasury Bonds), volatility targets (fully invested 60/40, value-weighted market, and stocks) and the various levered strategies considered in this paper. Unlevered risk parity has the highest Sharpe ratio (0.52), followed closely by U.S. Treasury Bonds (0.49). However, both exhibited low volatility and low excess return, making them unattractive as asset allocations for most investors.[26] Levered strategies are attractive as an asset allocation only if the Sharpe ratio survives leverage.

As shown in table 2.8, the Sharpe ratios of the levered strategies were all lower than the Sharpe ratios of their source portfolios. This highlights a fact that is well-known but often neglected: outside of an idealized setting, the Sharpe ratio is *not* leverage invariant.

In this chapter, we highlight two features of a levered strategy that contribute to the difference between its Sharpe ratio and the Sharpe ratio of its source portfolio. The first is transaction costs. Both leverage-induced trading costs and financing costs diminish Sharpe ratio; see equation (3.1). The second is the covariance term. Since the covariance term was negative in the examples considered in this chapter, it lowered the Sharpe ratios of the dynamically levered strategies relative to the Sharpe ratios of their source portfolios and comparably calibrated fixed levered strategies. However, as indicated in figure 2.2, the correlation between leverage and the return to the source portfolio, which is the driver of the covariance term, can be highly unstable at horizons of three to five years. So unless a leverage-seeking investor has a specific reason to believe this correlation will be positive over a particular period for a particular dynamically levered strategy, or unless he or she enjoys the coin-flip-like risk illustrated in 2.2, that investor may prefer a fixed leverage strategy.

[25] See section 2.5 for an analysis of the covariance term from the standpoint of volatility targeting. Had we made the unrealistic assumptions that financing was at the risk-free rate, and that trading costs were zero, the two FLT strategies would still have easily outperformed the UVT strategy.

[26] Of course, bonds are often used as one asset class in an asset allocation, such as 60/40 or the value-weighted market portfolio. 60/40 has been widely used as an asset allocation, and risk parity has been proposed as an alternative asset allocation; see, for example, Asness et al. (2012).

2.5 The Covariance Term, Revisited

The most novel part of our analysis is its focus on the covariance between leverage and excess borrowing return. In this section, we examine the covariance term from the standpoint of volatility targeting. We have already noted that leverage reduces the Sharpe ratio if the borrowing rate exceeds the risk-free rate, or if trading incurs costs. However, in a multi-period setting, leverage has an impact on Sharpe ratio even in the absence of those market frictions, via the covariance term. In order to focus on the covariance term, we make the highly unrealistic assumptions that borrowing is at the risk-free rate (i.e. $r^b = r^f$), which is fixed, and that trading costs are zero. We find that applying UVT leverage *does* change the Sharpe ratio, even under these assumptions.[27] Variable leverage, as used in UVT, is "an unintended market-timing strategy."[28]

Under these unrealistic assumptions, the excess return of the levered strategy is given by:

$$r^{\mathbf{L}} - r^f = \lambda \left(r^{\mathbf{S}} - r^f \right) \tag{2.9}$$

Suppose we pick a fixed volatility target V; then we must set $\lambda = \frac{V}{\text{volatility of source}}$. Thus, we have:

$$\begin{aligned}
&\text{Sharpe ratio of levered strategy} \\
&= \frac{E\left[r^{\mathbf{L}} - r^f\right]}{V} \\
&= \frac{E\left[\lambda\left(r^{\mathbf{S}} - r^f\right)\right]}{V} \\
&= \frac{E\left[\frac{V}{\text{volatility of source}} \times \left(r^{\mathbf{S}} - r^f\right)\right]}{V} \\
&= \frac{E\left[V \times \frac{r^{\mathbf{S}} - r^f}{\text{volatility of source}}\right]}{V} \\
&= E\left[\frac{r^{\mathbf{S}} - r^f}{\text{volatility of source}}\right] \\
&= E\left[r^{\mathbf{S}} - r^f\right] E\left[\frac{1}{\text{volatility of source}}\right] + \mathbf{C}ov\left(r^{\mathbf{S}} - r^f, \frac{1}{\text{volatility of source}}\right)
\end{aligned} \tag{2.10}$$

[27] This issue has been misunderstood in the published literature. For example, Asness et al. (2013) wrote, "Scaling the returns to any stable risk target (or not scaling them at all) cannot mathematically affect the Sharpe ratio, or the t-statistic of the alpha of our levered portfolios, because we are multiplying the return stream by a fixed constant." Their analysis conflated single-period models with multi-period models, and misstated the construction of the UVT_{MKT} strategy used in Asness et al. (2012).

[28] Asness et al. (2013) asserted that variable *volatility*, rather than variable leverage, is "an unintended market-timing strategy."

Formula (2.10) makes it clear that a covariance term will be present in the Sharpe ratio of any strategy that involves levering a source portfolio of variable volatility to a volatility target. Our empirical examples show that the covariance in formula (2.5) has a material effect on realized return and realized Sharpe ratio. It follows that the covariance in formula (2.10) has a material effect on realized return and realized Sharpe ratio. Recall that the leverage was especially volatile in the levered strategy $\mathrm{UVTB}_{\mathrm{STOCKS}}$, which levered a source portfolio of U.S. Treasury bonds to the volatility of stocks; even though the target volatility was constant, the leverage was very volatile precisely because the inverse of the volatility of bonds was high.

2.6 Concluding Remarks

In this chapter, we developed a platform that supports both backward-looking performance attribution and forward-looking investment decisions concerning levered strategies. Specifically, in formula (2.6), we expressed the difference between arithmetic expected return to a levered strategy portfolio and its source portfolio as a sum of four terms:

$$E\left[r^{\mathbf{L}}\right] = E\left[r^{\mathbf{S}}\right] + E\left[\lambda - 1\right] E\left[r^{\mathbf{S}} - r^{b}\right] + \mathbf{C}ov(\lambda, r^{\mathbf{S}} - r^{b}) - \left(E\left[r^{\mathbf{TCS}}\right] + E\left[r^{\mathbf{TCL}}\right]\right).$$

The first two terms, whose sum we have called *magnified source return*, are the ones that most easily come to mind in the context of a levered strategy. However, as we have shown empirically, other factors have a material effect on the cumulative return to a levered strategy. These include the covariance of leverage with the excess borrowing return, trading costs and compounding effects.

Formula (2.6) accounted for both the covariance term and transaction costs. However, it neglected the effect of compounding, which imposes a variance drag on cumulative return that is not captured in arithmetic expected return. If the levered strategy has high volatility, the variance drag may be substantial. Hence a more accurate decision rule depends on geometric expected return in formula (2.8):

$$G[r] \sim (1 + E[r])e^{-\frac{\mathbf{V}ar(r)}{2}} - 1.$$

We used formulas (2.6) and (2.8) to examine the realized performance of fixed leverage (FLT) strategies and two dynamically levered strategies: unconditional volatility targeting (UVT) and conditional volatility targeting (CVT). Some scholars have expressed the view that CVT strategies are poor alternatives to UVT strategies;[29] this view is not supported by the results reported in tables 2.4 and 2.6. In fact, it is the leverage that was implicitly determined by the volatility targets in $\mathrm{UVT}_{60/40}$ and $\mathrm{CVT}_{60/40}$, and not the volatility itself, that interacted with the return to the source portfolio to determine strategy performance. In our

[29] See Asness et al. (2013).

1929–2012 period, $\mathrm{CVT}_{60/40}$ outperformed $\mathrm{UVT}_{60/40}$. Future work is required to determine whether the sign of the covariance term might be predictable at longer horizons.

In the examples we considered, the cumulative effects of borrowing and trading costs, the variance drag and the covariance term offset much of the benefit of return magnification. Leverage, both fixed and dynamic, substantially lowered Sharpe ratios. In addition, dynamic leverage added noise to returns. Over our 84-year time horizon, fixed leverage strategies outperformed volatility-targeting strategies, and levered strategies had lower Sharpe ratios than their unlevered source portfolios.

Asness et al. (2012) argued that risk parity (levered to the volatility of the market) outperformed 60/40 over a long horizon;[30] our analysis does not support this.[31] Risk parity performed relatively well over the period 2008–2012, which featured Fed-supported interest rates that were extraordinarily low by historical standards. But that need not indicate how risk parity will perform in other regimes. Rising interest rates tend to raise the cost of funding a levered strategy and lower the prices of bonds in risk parity portfolios at the same time. Rising interest rates also have the potential to limit corporate profits and thereby exert downward pressure on equity prices. These considerations should be incorporated into any decision to lever low-risk portfolios when interest rates are unusually low.

[30]It is reasonable to compare the performance of the value-weighted market to risk parity levered to the volatility of the market, and to compare the performance of 60/40 to risk parity levered to the volatility of 60/40. However, it does not seem reasonable to compare the performance of 60/40 to that of risk parity levered to the volatility of the market; we are grateful to Patrice Boucher for this insight.

[31]See, however, the discussion in section 2.3.

Table 2.2: Performance Attribution

Sample Period: 1929-2012 Source: Risk Parity, Target: 60/40 r^b = 3M-EDR, with trading costs		$UVT_{60/40}$
Total Source Return (gross of trading costs)		**5.75**
Leverage	2.66	
Excess Borrowing Return	1.49	
Levered Excess Borrowing Return		3.97
Magnified Source Return		**9.72**
Volatility of Leverage	7.7212	
Volatility of Excess Borrowing Return	4.2219	
Correlation(Leverage,Excess Borrowing Return)	-0.0566	
Covariance(Leverage,Excess Borrowing Return)		-1.84
Source Trading Costs		-0.07
Leverage-Induced Trading Costs		-0.96
Total Levered Return (arithmetic)		**6.85**
Compounded Arithmetic Return (gross)	1.0707	
Variance Correction	0.9934	
Variance Drag		-0.48
Approximation Error		0.00
Total Levered Return (geometric)		**6.37**

Notes: Performance attribution of the realized geometric return of the levered strategy $UVT_{60/40}$ in terms of its source portfolio, risk parity, over the period January 1929–December 2012. The performance attribution was based on Formulas (2.6) and (2.8). Borrowing was at the Eurodollar deposit rate and trading costs were based on the linear model in appendix B.3. Arithmetic returns were estimated from monthly data and annualized by multiplication by 12; they are displayed in percent. Geometric returns are also displayed in percent and were annualized by $(1 + G[r])^{12} - 1$. Formulas corresponding to the words in the performance attribution are presented in table B.9.

Table 2.3: Historical Performance

Sample Period: 1929-2012 Source: Risk Parity, Target: 60/40 r^b = 3M-EDR	Arithmetic Total Return	Geometric Total Return	Average Leverage	Volatility	Arithmetic Excess Return	Sharpe Ratio	Skewness	Excess Kurtosis
60/40	8.18	7.77	1.00	11.58	4.69	0.40	0.19	7.44
Risk Parity	5.68	5.74	1.00	4.20	2.20	0.52	0.05	4.92
$\mathrm{UVT}_{60/40}$	6.85	6.37	3.66	11.54	3.37	0.29	-0.43	2.23

Notes: Annualized arithmetic and geometric returns, volatility and Sharpe ratio, of $\mathrm{UVT}_{60/40}$ (risk parity levered to an unconditional volatility target of 11.59%, the realized volatility of 60/40), the source portfolio (unlevered risk parity), and the volatility target (60/40) over the period 1929–2012. Arithmetic returns were estimated from monthly data and annualized by multiplication by 12; they are displayed in percent. Geometric returns are also displayed in percent and were annualized by $(1 + G[r])^{12} - 1$. Volatility was measured from monthly returns and annualized by multiplying by $\sqrt{12}$. Sharpe ratios were calculated using annualized excess return and annualized volatility.

Table 2.4: Performance Attribution

Sample Period: 1929-2012 Source: Risk Parity, Target: 60/40 r^b = 3M-EDR, with trading costs		$\mathrm{UVT}_{60/40}$		$\mathrm{FLT}_{60/40,\lambda}$		$\mathrm{FLT}_{60/40,\sigma}$		$\mathrm{CVT}_{60/40}$
Total Source Return (gross of trading costs)		**5.75**		**5.75**		**5.75**		**5.75**
Leverage	2.66		2.69		1.75		2.31	
Excess Borrowing Return	1.49		1.49		1.49		1.49	
Levered Excess Borrowing Return		3.97		4.02		2.61		3.45
Magnified Source Return		**9.72**		**9.77**		**8.37**		**9.20**
Volatility of Leverage	7.7212		0.0000		0.0000		5.0791	
Volatility of Excess Borrowing Return	4.2219		4.2219		4.2219		4.2219	
Correlation(Leverage,Excess Borrowing Return)	-0.0566		0.0000		0.0000		-0.0299	
Covariance(Leverage,Excess Borrowing Return)		-1.84		0.00		0.00		-0.64
Source Trading Costs		-0.07		-0.07		-0.07		-0.07
Leverage-Induced Trading Costs		-0.96		-0.51		-0.27		-0.93
Total Levered Return (arithmetic)		**6.85**		**9.19**		**8.03**		**7.56**
Compounded Arithmetic Return (gross)	1.0707		1.0959		1.0833		1.0783	
Variance Correction	0.9934		0.9881		0.9934		0.9926	
Variance Drag		-0.48		-0.91		-0.41		-0.53
Approximation Error		0.00		0.01		0.01		0.00
Total Levered Return (geometric)		**6.37**		**8.29**		**7.62**		**7.03**

Notes: Performance attribution of the realized geometric return of the levered strategies $\mathrm{UVT}_{60/40}$, $\mathrm{FLT}_{60/40,\lambda}$, $\mathrm{FLT}_{60/40,\sigma}$, and $\mathrm{CVT}_{60/40}$ in terms of their common source portfolio, risk parity, over the period January 1929–December 2012. $\mathrm{FLT}_{60/40,\lambda}$ had constant leverage 3.69, matching the average leverage of $\mathrm{UVT}_{60/40}$, while $\mathrm{FLT}_{60/40,\sigma}$ had constant leverage 2.75, chosen to match the volatility of $\mathrm{UVT}_{60/40}$. The performance attribution was based on Formulas (2.6) and (2.8). Borrowing was at the Eurodollar deposit rate and trading costs were based on the linear model in appendix B.3. Arithmetic returns were estimated from monthly data and annualized by multiplication by 12; they are displayed in percent. Geometric returns are also displayed in percent and were annualized by $(1+G[r])^{12}-1$. Formulas corresponding to the words in the performance attribution are presented in table B.9.

Table 2.5: Strategy Responses to Changes in Market Conditions

Trigger	Response: FLT	UVT	CVT
Increase in Target Volatility	no change	no change	↑ leverage
Increase in Source Volatility	no change	↓ leverage	↓ leverage
Increase in Price of Source	buy source	buy source	buy source

Table 2.6: Performance Attribution

Sample Period: 1929-2012 Source: Risk Parity, Targets: VW Market, 60/40 r^b = 3M-EDR, with trading costs		UVT_{MKT}		$UVT_{60/40}$		CVT_{MKT}		$CVT_{60/40}$
Total Source Return (gross of trading costs)		**5.75**		**5.75**		**5.75**		**5.75**
Leverage	3.71		2.66		2.58		2.31	
Excess Borrowing Return	1.49		1.49		1.49		1.49	
Levered Excess Borrowing Return		5.55		3.97		3.85		3.45
Magnified Source Return		**11.30**		**9.72**		**9.60**		**9.20**
Volatility of Leverage	9.9463		7.7212		5.3164		5.0791	
Volatility of Excess Borrowing Return	4.2219		4.2219		4.2219		4.2219	
Correlation(Leverage,Excess Borrowing Return)	-0.0566		-0.0566		-0.0321		-0.0299	
Covariance(Leverage,Excess Borrowing Return)		-2.37		-1.84		-0.72		-0.64
Source Trading Costs		-0.07		-0.07		-0.07		-0.07
Leverage-Induced Trading Costs		-1.40		-0.96		-1.13		-0.93
Total Levered Return (arithmetic)		**7.45**		**6.85**		**7.68**		**7.56**
Compounded Arithmetic Return (gross)	1.0771		1.0707		1.0796		1.0783	
Variance Correction	0.9891		0.9934		0.9872		0.9926	
Variance Drag		-0.92		-0.48		-1.11		-0.53
Approximation Error		0.00		0.00		-0.05		0.00
Total Levered Return (geometric)		**6.53**		**6.37**		**6.52**		**7.03**

Notes: Performance attribution of the realized geometric return of the levered strategies UVT_{MKT}, $UVT_{60/40}$, CVT_{MKT} and $CVT_{60/40}$ in terms of their common source portfolio, risk parity, over the period January 1929–December 2012. The performance attribution was based on Formulas (2.6) and (2.8). Borrowing was at the Eurodollar deposit rate and trading costs were based on the linear model in appendix B.3. Arithmetic returns were estimated from monthly data and annualized by multiplication by 12; they are displayed in percent. Geometric returns are also displayed in percent and were annualized by $(1+G[r])^{12}-1$. Formulas corresponding to the words in the performance attribution are presented in table B.9.

Table 2.7: Performance Attribution

Sample Period: 1929-2012 Source: Bonds, Target: Stocks r^b = 3M-EDR, with trading costs		$\text{UVTB}_{\text{STOCKS}}$		$\text{FLTB}_{\text{STOCKS},\lambda}$		$\text{FLTB}_{\text{STOCKS},\sigma}$
Total Source Return (gross of trading costs)		**5.08**		**5.08**		**5.08**
Leverage	6.43		6.49		4.80	
Excess Borrowing Return	0.82		0.82		0.82	
Levered Excess Borrowing Return		5.29		5.34		3.95
Magnified Source Return		**10.37**		**10.42**		**9.03**
Volatility of Leverage	17.4159		0.0000		0.0000	
Volatility of Excess Borrowing Return	3.2711		3.2711		3.2711	
Correlation(Leverage,Excess Borrowing Return)	-0.0742		0.0000		0.0000	
Covariance(Leverage,Excess Borrowing Return)		-4.23		0.00		0.00
Source Trading Costs		0.00		0.00		0.00
Leverage-Induced Trading Costs		-2.59		-1.62		-0.93
Total Levered Return (arithmetic)		**3.55**		**8.80**		**8.10**
Compounded Arithmetic Return (gross)	1.0361		1.0916		1.0841	
Variance Correction	0.9820		0.9707		0.9823	
Variance Drag		-1.80		-2.84		-1.61
Approximation Error		-0.05		-0.02		0.00
Total Levered Return (geometric)		**1.70**		**5.93**		**6.49**

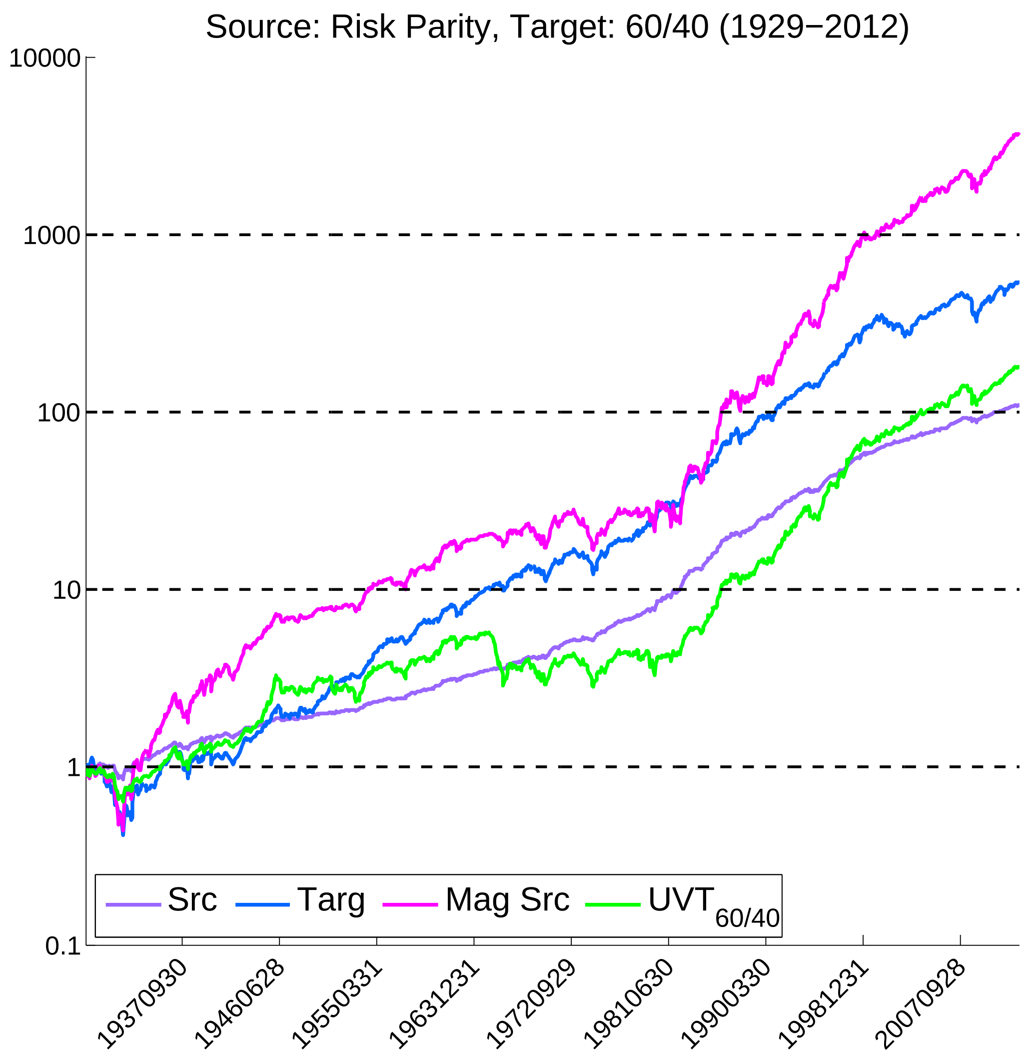
Notes: Performance attribution of the realized geometric return of the levered strategies $\text{UVTB}_{\text{STOCKS}}$, $\text{FLTB}_{\text{STOCKS},\lambda}$ and $\text{FLTB}_{\text{STOCKS},\sigma}$ in terms of their common source portfolio, U.S. Treasury bonds, over the period January 1929–December 2012. $\text{UVTB}_{\text{STOCKS}}$ was levered to the volatility of stocks (18.93%) over the period 1929–December 2012. $\text{FLTB}_{\text{STOCKS},\lambda}$ had fixed leverage 8.72, equal to the average leverage of $\text{UVTB}_{\text{STOCKS}}$; $\text{FLTB}_{\text{STOCKS},\sigma}$ had fixed leverage and the same volatility 22.47% as $\text{UVTB}_{\text{STOCKS}}$. The performance attribution was based on Formulas (2.6) and (2.8). Borrowing was at the Eurodollar deposit rate and trading costs were based on the linear model in appendix B.3. Arithmetic returns were estimated from monthly data and annualized by multiplication by 12; they are displayed in percent. Geometric returns are also displayed in percent and were annualized by $(1 + G[r])^{12} - 1$. Formulas corresponding to the words in the performance attribution are presented in table B.9.

Table 2.8: Historical Performance

Sample Period: 1929-2012 Source: Various, Target: Various r^b = 3M-EDR	Arithmetic Total Return	Geometric Total Return	Average Leverage	Volatility	Arithmetic Excess Return	Sharpe Ratio	Skewness	Excess Kurtosis
60/40	8.18	7.77	1.00	11.58	4.69	0.40	0.19	7.44
VW Market	8.12	7.24	1.00	14.93	4.63	0.31	0.61	14.39
Stocks	10.43	9.00	1.00	18.93	6.95	0.37	0.18	7.46
Risk Parity	5.68	5.74	1.00	4.20	2.20	0.52	0.05	4.92
Bonds	5.08	5.14	1.00	3.26	1.59	0.49	0.03	4.74
$\text{UVT}_{60/40}$	6.85	6.37	3.66	11.54	3.37	0.29	-0.43	2.23
$\text{FLT}_{60/40,\lambda}$	9.19	8.29	3.69	15.53	5.70	0.37	-0.01	4.78
$\text{FLT}_{60/40,\sigma}$	8.03	7.62	2.75	11.57	4.54	0.39	0.00	4.80
$\text{CVT}_{60/40}$	7.56	7.03	3.31	12.22	4.07	0.33	-0.41	7.13
UVT_{MKT}	7.45	6.53	4.71	14.88	3.97	0.27	-0.44	2.23
CVT_{MKT}	7.68	6.52	3.58	16.13	4.19	0.26	-0.75	15.62
$\text{UVTB}_{\text{STOCKS}}$	3.55	1.70	7.43	19.10	0.07	0.00	-0.55	4.75
$\text{FLTB}_{\text{STOCKS},\lambda}$	8.80	5.93	7.49	24.47	5.31	0.22	-0.08	4.68
$\text{FLTB}_{\text{STOCKS},\sigma}$	8.10	6.49	5.80	18.95	4.61	0.24	-0.07	4.66

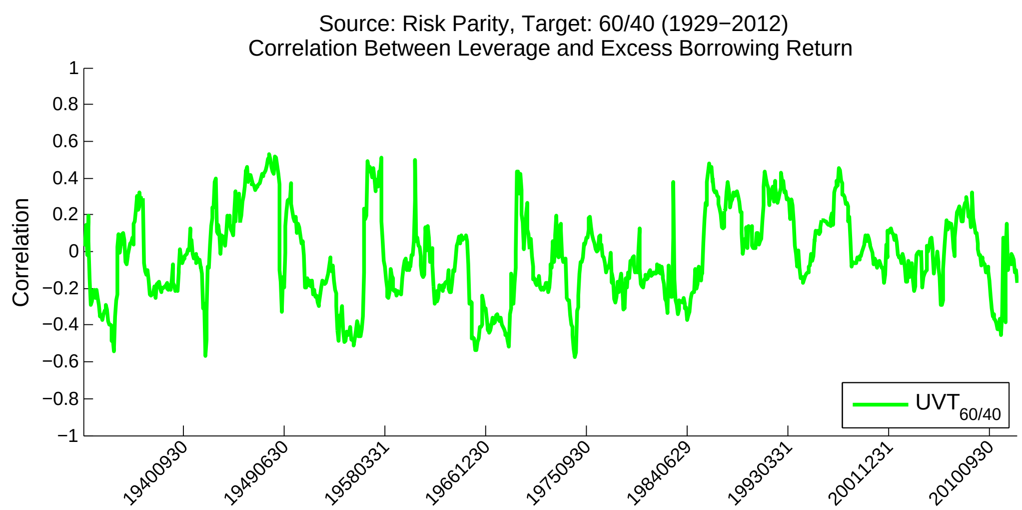
Notes: Annualized arithmetic and geometric returns, volatility and Sharpe ratio, of the source portfolios (unlevered risk parity and U.S. Treasury bonds, volatility targets (fully invested 60/40, value-weighted market, and stocks) and the various levered strategies considered in this paper, over the period 1929—2012. Arithmetic returns were estimated from monthly data and annualized by multiplication by 12; they are displayed in percent. Geometric returns are also displayed in percent and were annualized by $(1 + G[r])^{12} - 1$. Volatility was measured from monthly returns and annualized by multiplying by $\sqrt{12}$. Sharpe ratios were calculated using annualized excess arithmetic return and annualized volatility.

Figure 2.1: Cumulative Returns to Strategies over the Long Sample



Magnified source return (in magenta) and realized cumulative return (in light green) for $UVT_{60/40}$ (risk parity unconditionally levered to a target volatility of 11.59%) over the period 1929–2012. For comparison, we also plot the realized cumulative return of the volatility target (60/40 fixed mix, in blue) and the source (fully invested risk parity, in lavender). Magnified source return is an idealized return that cannot be achieved in practice; the curve depicts what we would earn if we achieved a geometric return equal to the arithmetic magnified source return.

Figure 2.2: Leverage and Excess Borrowing Return Correlation



Correlation of excess borrowing return and leverage for $UVT_{60/40}$, risk parity levered to match the realized volatility of 60/40 fixed mix over the period 1929–2012. Correlation was computed from monthly data using a trailing 36-month window.

Chapter 3

Risk Parity

The familiar disclaimer that *past performance is not a guarantee of future returns* highlights the fact that a particular investment strategy may work well in some periods and poorly in other periods, limiting the inference that can be drawn from past returns.

The concern is heightened when a proposed investment strategy is backtested using historic data. Consider an investment strategy that can be pursued today using readily available securities. If those securities were not available in the past, then the strategy has no true antecedent. Backtesting must be done using proxies for the securities, and the choice of proxies can have a direct effect on measured returns. In addition, the introduction of new securities can have an indirect effect; a strategy that seems to have been profitable in the past might have been less profitable if the new securities had been available and made the strategy accessible to a broader class of investors. The matter is confounded by the specific attributes of the backtesting period, concerns about statistical significance, and a plethora of metrics used by investors to evaluate strategy performance.

In this chapter, we consider these issues by carefully examining the historical performance of four simple strategies based on two asset classes: US Equity and US Treasury Bonds.[1] Our study includes a market or value weighted portfolio, which is the optimal risky portfolio in the Capital Asset Pricing Model (CAPM), and a 60/40 mix, which is popular with pension funds and other long horizon investors.

Our study also includes two risk parity strategies. Risk parity attempts to equalize risk contributions across asset classes, Early formulations of risk parity are in Lörtscher (1990) and Kessler and Schwarz (1996).[2] Risk parity has been popular since the 2008 financial crisis, as frustrated investors have struggled to meet return targets by levering low-risk or

[1] Our simple, two asset class strategies, which involve no market timing and no security selection, can be used as benchmarks to evaluate more complex strategies that are used in practice.

[2] In Lörtscher (1990) and Kessler and Schwarz (1996), risk parity strategies are known as "equal risk benchmarks."

low-beta assets, and it is sufficiently mainstream to be featured in the *Wall Street Journal*.[3] A diverse collection of risk parity strategies can be constructed by varying asset classes, grouping schemes, and risk estimates.[4]

An essential element of risk parity is leverage, and it is leverage that distinguishes the two risk parity strategies in our study. An unlevered risk parity strategy tends to have relatively low risk and consequently relatively low expected return, so a risk parity strategy must be levered in order to have even a remote chance of achieving a typical return target.[5] The notion that levering a low-risk portfolio might be worthwhile dates back to Black, Jensen and Scholes (1972), which provides empirical evidence that the risk-adjusted returns of low-beta equities are higher than what is predicted by the CAPM. Black (1972) introduces a zero-beta portfolio, which is considered by some to be the antecedent of risk parity. Nearly four decades later, Frazzini and Pedersen (2011) developed a compelling theory of leverage aversion in which risk parity emerges as a dominant strategy, and this dominance is supported by the empirical study in Asness, Frazzini and Pedersen (2012). However, our results do not support this dominance.

We find that performance depends materially on the backtesting period. For example, in our 85-year Long Sample, 1926–2010, if we assume borrowing was at the risk-free rate[6] and there were no trading costs, the levered risk parity strategy had the highest cumulative return. However the outperformance was not uniform across relatively long sub-periods. For example, in our 37-year Post-War Sample, 1946–1982, both the value weighted and 60/40 strategies had higher cumulative returns than the risk parity strategies did.

We find that performance depends materially on assumptions made about market frictions. Since we do not know how the availability of modern financing would have affected markets during the early part of our study period,[7] we extrapolate borrowing costs from

[3]Dagher (February 6, 2012) discusses the long-term outlook for risk parity strategies.

[4]For example Qian (2005) considers the implications of including asset correlations in risk parity weights. Chaves, Hsu, Li and Shakernia (2011) consider a broader collection of asset classes, and they also consider risk parity in the context of other low-risk strategies.

[5]There is a large and growing literature on low-risk investing. Sefton, Jessop, Rossi, Jones and Zhang (2011) give a broad discussion of the topic, and Scherer (2011) attributes the empirically observed out-performance of the market by a particular low-risk (minimum variance) strategy to Fama-French factors. Cowan and Wilderman (2011) provide a rational explanation for the low-risk anomaly and Baker, Bradley and Wurgler (2011) provide a behavioral explanation. Clarke, de Silva and Thorley (2011) analyze the connection between minimum variance and low beta strategies.

[6]Throughout this chapter, the risk-free rate is proxied by the 90-Day T-Bill Rate.

[7] For a liquid asset class such as US Treasury bonds, futures may be the cheapest way to finance the levered position. However, US Treasury futures have been traded in a liquid market only since the 1980s. So it is impossible to conduct a fully empirical study of risk parity that begins early in the twentieth century because we don't know how a futures-financed risk parity strategy would have performed during the Great Depression. We can instead estimate what it would have cost to finance the leverage through more conventional borrowing, but small differences in assumptions about the cost of borrowing have major effects on the estimated returns of a levered risk parity strategy, precisely because the strategy involves such a high degree of leverage. Moreover, because the introduction of liquid US Treasury futures markets presumably

recent experience, and we base trading costs on conventional wisdom. We find that market frictions were a substantial drag on performance of the levered risk parity strategy. For example, in our 85-year Long Sample, 1926–2010, after adjusting for transaction costs,[8] both the value weighted and 60/40 strategies had higher cumulative returns than the levered risk parity strategy did. In other words, the ranking based on cumulative return was reversed after adjustment for market frictions. This reversal may be explained by the high degree of leverage in the levered risk parity strategy. The ranking based on cumulative return in Asness, Frazzini and Pedersen (2012) is also reversed. This reversal may be explained by the adjustment for market frictions, and by the fact that the Asness, Frazzini and Pedersen (2012) strategy contains lookahead bias, and is therefore uninvestable.

We find that a statistically significant risk premium may be far from a guarantee of outperformance in practical situations. Under the unrealistic, but nevertheless widely adopted, assumption that the underlying processes possess some strong form of stationarity, the high volatility of security returns poses two closely related practical problems:

- The confidence intervals on the returns of a strategy are very wide, even with many decades of data. Thus, it is rarely possible to demonstrate with conventional statistical significance that one strategy dominates another.

- Even if we were reasonably confident that one strategy achieved higher expected returns than another without incurring extra risk, it would be entirely possible for the weaker strategy to outperform over periods of several decades, certainly beyond the investment horizon of most individuals and even perhaps of institutions like pension funds or endowments.

We find that performance depends on the measure. Over the Long Sample, unlevered risk parity had the highest Sharpe Ratio and the lowest expected return. When unlevered risk parity was levered to have the same volatility as the value weighted portfolio, transaction costs reduced its Sharpe Ratio and its cumulative return was less than the return of the 60/40 and value weighted strategies.[9] Therefore, the empirical observation that levered risk parity outperforms the market in an idealized setting may be explained, at least in part, by the fact that an idealized setting does not include market frictions.

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reduced the cost of financing a levered risk parity, it may have induced changes in asset returns that would have tended to offset the savings achieved through lower financing costs.

[8]Specifically, borrowing is at the 3-Month Euro-Dollar Deposit Rate starting in 1971, and is equal to the risk-free rate plus sixty basis points before 1971. Turnover-induced trading costs are 1% during the period 1926–1955, .5% during the period 1956–1970 and .1% during the period 1971–2010.

[9]Chaves, Hsu, Li and Shakernia (2011), comment that realistic borrowing costs might affect the Sharpe ratio: "... it is unclear whether their [unlevered risk parity] Sharpe ratios would remain the same after financing costs."

3.1 Study Outline and Rationale for Some of Our Assumptions

Strategies: We evaluate four strategies based on two asset classes: US Equity and US Treasury Bonds. The strategies are value weighted, 60/40, unlevered risk parity and levered risk parity. Unlevered risk parity is a fully invested strategy weighted so that ex post risk contributions coming from the asset classes are equal. If we lever this strategy to match the ex post volatility of the value weighted portfolio we obtain levered risk parity. Weights in the risk parity strategies depend on volatility estimates, which are based on three-year rolling windows. The strategies are rebalanced monthly. The data and formulas required to replicate our results are in Appendices C.1 and C.2.

Study Periods: We evaluate the four strategies over an 85-year Long Sample, 1926–2010, and four sub-periods. The 20-year Pre-1946 Sample, 1926–1945, which included the Great Depression and World War II, was plagued by deflationary shocks and inflationary spikes. Equity markets were uneven during the 37-year Post-War Sample, 1946–1982. This period included spikes in inflation and high interest rates that translated into poor bond performance. The 18-year Bull Market Sample, 1983–2000, included a huge bond rally and the game-changing emergence of the technology industry. The ten-year period that began with the bursting of the DotCom bubble felt turbulent, although it was much calmer than the initial years of the study period.

Transaction Costs: We evaluate the four strategies in each period under three sets of assumptions about transaction costs. The base case assumes borrowing was at the risk-free rate and turnover-induced trading incurred no penalty. The middle case assumes borrowing was at the 3-Month Euro-Dollar Deposit Rate starting in 1971, and was at the risk-free rate plus sixty basis points before 1971. The rationale for this stems from Naranjo (2009), which concludes that investors employing futures borrow at LIBOR rates on average. Since LIBOR rates are available beginning only in 1987, Eurodollar deposit rates are available beginning in 1971, and 3-Month LIBOR and 3-Month Euro-Dollar Deposit Rates track one another closely over the period of overlap, we opted to use Eurodollar deposit rates in our study. The average spread of Eurodollar deposit rates over the risk-free rate during the period 1971–2010 is 100 basis points, so we conservatively assumed a borrowing rate of 60 basis points above the risk-free rate during the 1926–1970 period.

The final case retains borrowing assumptions from the middle case, and adds turnover-induced trading costs of 1% during the period 1926–1955, .5% during the period 1956–1970 and .1% during the period 1971–2010. The details of our turnover estimates and associated penalties are in Appendix C.3.

Statistical Significance: Confidence in parameters and strategy outperformance is estimated with a non-parametric bootstrap that is described in Appendix C.4.

Connection to Existing Literature: The data and three of our four strategies: value weighted, 60/40 and unlevered risk parity, are identical to the data and similarly named strategies in the Long Sample in Asness, Frazzini and Pedersen (2012), and our performance estimates match to a high degree of precision. Unlike the levered risk parity strategy in Asness, Frazzini and Pedersen (2012), ours is conditional: it is rebalanced so that its ex post volatility over a three-year window matches the ex post volatility of the value weighted strategy at each rebalancing date. The levered risk parity strategy in Asness, Frazzini and Pedersen (2012) is unconditional: it employs a constant scale factor chosen to match the ex post volatility of the value weighted strategy over the entire study period. Comparing Asness, Frazzini and Pedersen (2012, Figure 1) to our figure 3.1, the cumulative return of the unconditional (and uninvestable) levered risk parity strategy was roughly double the cumulative return of the conditional version over the Long Sample.[10]

3.2 The Specific Start and End Dates of a Backtest Can Have a Material Effect on the Results

Figure 3.1 shows cumulative returns to the four strategies over the period 1926–2010. Levered risk parity had the highest return by a factor of three. However, the performance was uneven, as shown in figure 3.2, where the eight-and-a-half decade study period is broken into four substantial sub-periods.

On the basis of cumulative return, levered risk parity prevailed during the the Pre-1946 Sample and the Last 10 Years. Despite its relatively low volatility, even unlevered risk parity beat the value weighted and 60/40 strategies in the most recent period. During the post-war period from 1946 to 1982, both the 60/40 and value weighted strategies outperformed risk parity. Between 1982 and 2000, levered risk parity, 60/40 and value weighted strategies tied for first place.

[10] Asness, Frazzini and Pedersen (2012, page 58) find that their unconditional levered risk parity, when financed at LIBOR rates, outperformed 60/40 and value weighted strategies over the Long Sample. They assert that they "obtained similar results by choosing k_t [the factor that scales the strategy to the target volatility level] to match the conditional volatility of the benchmark at the time of portfolio formation." We find that conditional risk parity performs substantially less well than unconditional risk parity, and underperforms 60/40 in the Long Sample when realistic borrowing and trading costs are taken into effect. We also find that unconditional risk parity and 60/40 are virtually tied in the Long Sample when realistic borrowing, or borrowing and trading, costs are taken into account.

3.3 Transaction Costs Can Negate Apparent Outperformance

3.3.1 Borrowing Costs

In the studies discussed in section 3.2, we financed the levered risk parity strategy at the 90-Day T-Bill Rate, but that is not possible in practice. The studies in Naranjo (2009) indicate that in the most recent decade, LIBOR is a more realistic estimate of the implicit interest rate at which investors can lever using futures. Because it is available over a longer period, we use the US 3-Month Euro-Dollar Deposit Rate as a proxy for LIBOR.[11] We repeat the studies in section 3.2 replacing the 90-Day T-Bill Rate with the 3-Month Euro-Dollar Deposit Rate rate starting in 1971, and using 90-Day T-Bill Rate plus 60 basis points in the prior period 1926–1970. Because the levered risk parity strategy involves substantial leverage, the effect of this relatively small change in borrowing rate on the return is magnified.

In this experiment, the 60/40 strategy had a slightly higher return than levered risk parity over the long horizon, 1926–2010. This is shown in figure 3.3. This reverses the ranking based on cumulative return when borrowing is at the risk-free rate, and it reverses the ranking based on cumulative return in Asness, Frazzini and Pedersen (2012).

The breakdown in figure 3.4 is consistent with the assertion that levered risk parity outperforms in turbulent periods and not otherwise. But the data are insufficient to decide on a purely statistical basis whether this assertion has any credence.

3.3.2 Trading Costs

Value weighted strategies require rebalancing only in response to a limited set of events, for example, new issues and redemptions of bond and shares. The risk parity and 60/40 strategies require additional rebalancing in response to price changes, and hence, they have higher turnover rates. Since we do not have data on new issues or redemptions, and since these should affect the four portfolios in a similar way, we measure the turnover in the risk parity and 60/40 strategies resulting from price changes.[12] As suggested by figure 3.5, leverage exacerbates turnover, so the trading costs for the levered risk parity are much higher than they are for the unlevered risk parity and 60/40 strategies. However, the data required to determine the precise relationship between turnover and trading costs are not available. So we estimate.[13]

Figure 3.6 shows the cumulative return to the four strategies over the long horizon. The

[11] Over the period when the US 3-Month Euro-Dollar Deposit Rate and 3-month LIBOR are both available, they track each other very closely, with LIBOR being about 10 basis points higher on average.

[12] The details of our turnover estimates are in Appendix C.3.

[13] We assume trading costs are 1% during the period 1926–1955, .5% during the period 1956–1970 and .1% during the period 1971–2010.

levered risk parity strategy is financed at the 3-Month Euro-Dollar Deposit Rate. Turnover-induced trading costs are incorporated in the returns to the 60/40 and risk parity strategies. From the perspective of return, 60/40 is the dominant strategy once again. This time, the value weighted and levered risk parity strategies finish in a tie. Figure 3.7 shows the breakdown into sub-periods.

3.4 Statistical Significance of Findings Needs to be Assessed

Because the volatility of asset return is substantially larger than its expected value, it is difficult to achieve statistical significance in a comparison of investment strategies, even over periods of decades. Table 3.1 presents P-values[14] for these comparisons. Disregarding trading costs and assuming borrowing was at the risk-free rate, the (annualized monthly arithmetic) mean return of levered risk parity exceeded that of 60/40 in the 85-year Long Sample by 210 basis points, and the result is statistically significant ($P = 0.03$). However, 60/40 was somewhat less volatile than levered risk parity; taking this into account, the alpha for levered risk parity minus 60/40 just fails to be significant ($P = 0.06$).

Once we take account of borrowing costs that exceed the risk-free rate, the annualized return of levered risk parity exceeded that of 60/40 by only 29 basis points, and is nowhere close to being statistically significant ($P = 0.40$).[15] The alphas were essentially tied.

If we also take into account trading costs, 60/40 beat levered risk parity, but the results are not statistically significant. Keep in mind that we are using more than eight decades of data in this analysis, but fail to find statistical significance.

Let's turn the problem around. Suppose we ignore trading costs and assume we can borrow at the risk-free rate. Suppose that, based on our point estimate from our Long Sample, we assume that the expected return of levered risk parity exceeds that of 60/40 by exactly 210 basis points. A bootstrap estimate of the probability that 60/40 will do better than levered risk parity over the next 20 years is 26.8%; over the next 50 years, it is still 17.5%. So even if you ignore borrowing and trading costs, 60/40 has a substantial probability of beating levered risk parity over the next 20 years and the next 50 years.

[14]In tests of statistical significance tests, a P-value is the probability of obtaining a test statistic at least as extreme as the one that was actually observed, assuming that the null hypothesis is true. In table 3.1, the null hypothesis takes one of two forms: either expected return is zero or regression alpha is zero.

[15]There is an apparent conflict between the information in the second panel of table 3.1 and the information in figure 3.3. Table 3.1 shows that if we take account of borrowing costs that exceed the risk-free rate but do not adjust for trading costs due to turnover, levered risk parity outperforms 60/40 by a (statistically insignificant) 29 basis points. Figure 3.3 shows that under the same assumptions, 60/40 outperforms levered risk parity over the 85-year period between 1926–2010. Table 3.1 reports the arithmetic mean of the monthly returns, which does not handle compounding correctly. Figure 3.3 presents the cumulative returns to the strategies over time, which would correspond to the geometric mean of the monthly returns.

Of course, even if you do take account of borrowing and trading costs, levered risk parity has a substantial probability of beating 60/40 over the next 20 years and the next 50 years.

3.5 Risk Profiles

A thorough evaluation of the four investment strategies involves risk as well as return. In this section, we consider the realized Sharpe ratios of the four strategies. Figure 3.8 shows the strategy Sharpe ratios over 1926–2010, and sub-period Sharpe ratios are in figure 3.9. These figures indicate that *unlevered risk parity* has the highest realized Sharpe ratio, with 60/40 coming second.[16] In the Capital Asset Pricing Model (CAPM), the value weighted portfolio uniquely maximizes the Sharpe ratio over the feasible set of portfolios with holdings limited to the risky assets. So the results in figure 3.8 suggest that the CAPM may not hold.[17]

A consideration that does not depend on the CAPM is the difference between the borrowing rate and the risk-free rate. When that difference is zero, an investor should hold a weighted combination of the risk-free asset and the risky portfolio with the maximum Sharpe ratio. The weights, can be positive, negative or zero. A weighted combination of this type maximizes return for given levels of risk.

However, in the more realistic case when the borrowing rate is higher than the risk-free rate, leverage diminishes the Sharpe ratio. Specifically, for a portfolio with leverage $\lambda > 1$,

$$S_L = S_U - \left(\frac{\lambda - 1}{\lambda}\right)\left(\frac{r_b - r_f}{\sigma}\right) \tag{3.1}$$

where S_L and S_U are the Sharpe ratios of the otherwise equivalent levered and unlevered portfolios, r_f is the risk-free rate, r_b is the borrowing rate, and σ is the volatility of the unlevered portfolio. For large leverage,

$$S_L \approx S_U - \frac{r_b - r_f}{\sigma}. \tag{3.2}$$

When the borrowing rate exceeds the risk-free rate, the efficient frontier is composed of three components, a line segment, an arc of parabola, and a half-line, as depicted schematically in figure 3.10. Note that the Sharpe ratio of a levered portfolio on the efficient frontier,

[16] The Sharpe ratios of the levered and unlevered risk parity strategies do not agree, even when borrowing is at the risk-free rate and we ignore trading costs. This is because the leverage is dynamic. The leverage ratio is chosen at each monthly rebalancing so that the conditional ex post volatilities of the levered risk parity and value weighted strategies match. If the levered risk parity strategy were constructed instead with fixed leverage, if borrowing were at the risk-free rate, and if there were no trading costs, the levered risk parity strategy would have the same Sharpe ratio as the unlevered risk parity strategy. Note that fixed leverage is not the same as the unconditional leverage in Asness, Frazzini and Pedersen (2012)

[17] Markowitz (2005) discusses a simple paradigm where leverage constraints render the market portfolio inefficient in an idealized setting.

which is given by formula 3.1, is equal to the slope of the line connecting the portfolio to the risk-free portfolio.

Why did the levered risk parity strategy in Asness, Frazzini and Pedersen (2012) outperform the others after adjusting for financing costs in excess of the risk-free rate, while an analogous adjustment to our levered risk parity strategy caused it to underperform? Asness, Frazzini and Pedersen (2012) match the Long Sample ex post volatility of the levered risk parity to the Long Sample ex post volatility of the value weighted strategy. Of course, this volatility cannot be known in advance, so the levered risk parity strategy in Asness, Frazzini and Pedersen (2012) is not investable.

Table 3.1 displays standard statistics on the four strategies. The best-performing strategy depends on how an investor weights different risk and performance measures. For example, when one has positive skewness, high kurtosis may be desirable, and this combination occurs for the valued weighted and 60/40 strategies. Levered risk parity exhibits negative skewness and high kurtosis. This bad combination can lead to de-leveraging costs, which could further degrade the performance of levered risk parity, but are beyond the scope of this chapter. These observations suggest an alternative to leverage aversion as an explanation for the performance of the frictionless version of levered risk parity: perhaps there is a premium for taking on severe downside risk.

3.6 Concluding Remarks

> When the experiments are done, we still have to decide what to believe.
> - Jonah Lehrer

Strategy evaluation is an important part of the investment process. However, since most strategies do not have true antecedents over long horizons, it is generally not possible to construct fully empirical backtests. Therefore, it is important to evaluate a strategy as broadly as possible—over periods of different length and in different market environments. It is essential to account for market frictions, to keep track of the assumptions underlying extrapolations, to estimate statistical significance, and to interpret results in an economic framework.

In this chapter, we examined a risk parity strategy of the type considered by pension funds, endowments and other long horizon investors who turn to leverage in an attempt to elevate return in a challenging market. Over the 85-year horizon between 1926 and 2010, the levered risk parity strategy we implemented returned substantially more than unlevered risk parity, a 60/40 fixed mix, and a value weighted portfolio. However, there are important caveats. First, levered risk parity underperformed during a relatively long sub-period: the 37-year Post-War Sample, 1946–1982. Second, transaction costs negated the gains over the full 85-year horizon, 1926–2010. Third, return is but one measure of performance. On the basis of risk-adjusted return, or realized Sharpe ratio, unlevered risk parity dominated the study. Other performance measures might lead to different conclusions.

Compelling economic theories of leverage aversion, such as the one in Frazzini and Pedersen (2011), give credence to the idea that levered risk parity may outperform the market over long horizons. However, there are dissenting voices, such as Sullivan (2010), which are also compelling. The studies in this chapter suggest that risk parity may be a preferred strategy under certain market conditions, or with respect to certain yardsticks. But any inference from our results must take account of the assumptions we made, and the fact that a study over any horizon, even a long one, is a single draw from a random distribution.

Table 3.1: Risk Parity vs. the Market vs. 60/40 (Historical Performance)

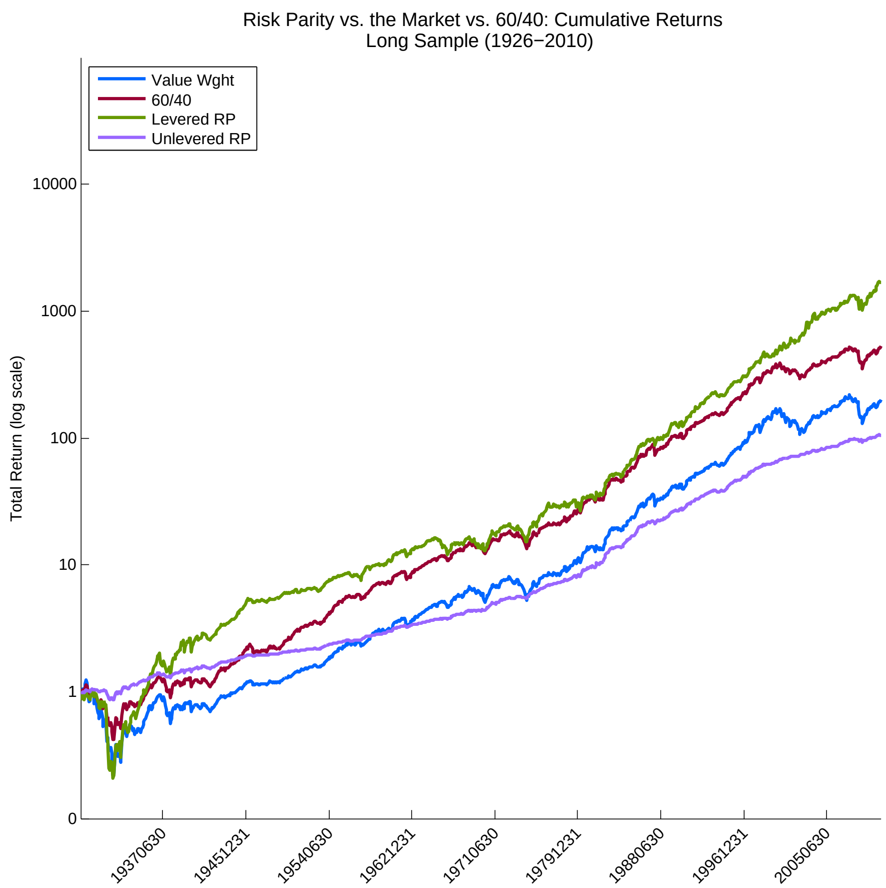
Panel A: Long Sample Stocks and Bonds, 1926-2010 Base Case	Excess Return	P-value Excess Return	Alpha	P-value Alpha	Volatility	Sharpe Ratio	Skewness	Excess Kurtosis
CRSP Stocks	6.93	0.00			19.05	0.36	0.18	7.44
CRSP Bonds	1.53	0.00			3.28	0.47	0.03	4.74
Value Weighted Portfolio	4.03	0.01			15.04	0.27	0.42	13.58
60/40 Portfolio	4.77	0.00			11.67	0.41	0.20	7.42
Risk Parity (unlevered)	2.21	0.00	1.36	0.00	4.24	0.52	0.07	4.80
Risk Parity (levered)	6.87	0.00	3.53	0.00	16.25	0.42	-0.58	15.54
Risk Parity (levered) minus Val Wght	2.84	0.01	3.53	0.00	10.73	0.26	-0.51	12.42
Risk Parity (levered) minus 60/40	2.10	0.03	1.81	0.06	10.11	0.21	-1.08	13.58

Panel B: Long Sample Stocks and Bonds, 1926-2010 Adjusted for 3M-EDR	Excess Return	P-value Excess Return	Alpha	P-value Alpha	Volatility	Sharpe Ratio	Skewness	Excess Kurtosis
CRSP Stocks	6.93	0.00			19.05	0.36	0.18	7.44
CRSP Bonds	1.53	0.00			3.28	0.47	0.03	4.74
Value Weighted Portfolio	4.03	0.01			15.04	0.27	0.42	13.58
60/40 Portfolio	4.77	0.00			11.67	0.41	0.20	7.42
Risk Parity (unlevered)	2.21	0.00	1.36	0.00	4.24	0.52	0.07	4.80
Risk Parity (levered)	5.06	0.00	1.70	0.07	16.29	0.31	-0.62	15.47
Risk Parity (levered) minus Val Wght	1.03	0.19	1.70	0.07	10.72	0.10	-0.57	12.50
Risk Parity (levered) minus 60/40	0.29	0.41	-0.02	0.51	10.11	0.03	-1.15	13.68

Panel C: Long Sample Stocks and Bonds, 1926-2010 Adjusted for 3M-EDR and Trading Costs	Excess Return	P-value Excess Return	Alpha	P-value Alpha	Volatility	Sharpe Ratio	Skewness	Excess Kurtosis
CRSP Stocks	6.93	0.00			19.05	0.36	0.18	7.44
CRSP Bonds	1.53	0.00			3.28	0.47	0.03	4.74
Value Weighted Portfolio	4.03	0.01			15.04	0.27	0.42	13.58
60/40 Portfolio	4.66	0.00			11.67	0.40	0.19	7.39
Risk Parity (unlevered)	2.14	0.00	1.29	0.00	4.24	0.50	0.06	4.80
Risk Parity (levered)	4.15	0.01	0.79	0.24	16.29	0.25	-0.66	15.39
Risk Parity (levered) minus Val Wght	0.11	0.47	0.79	0.24	10.75	0.01	-0.67	13.06
Risk Parity (levered) minus 60/40	-0.51	0.67	-0.81	0.77	10.13	-0.05	-1.22	13.98

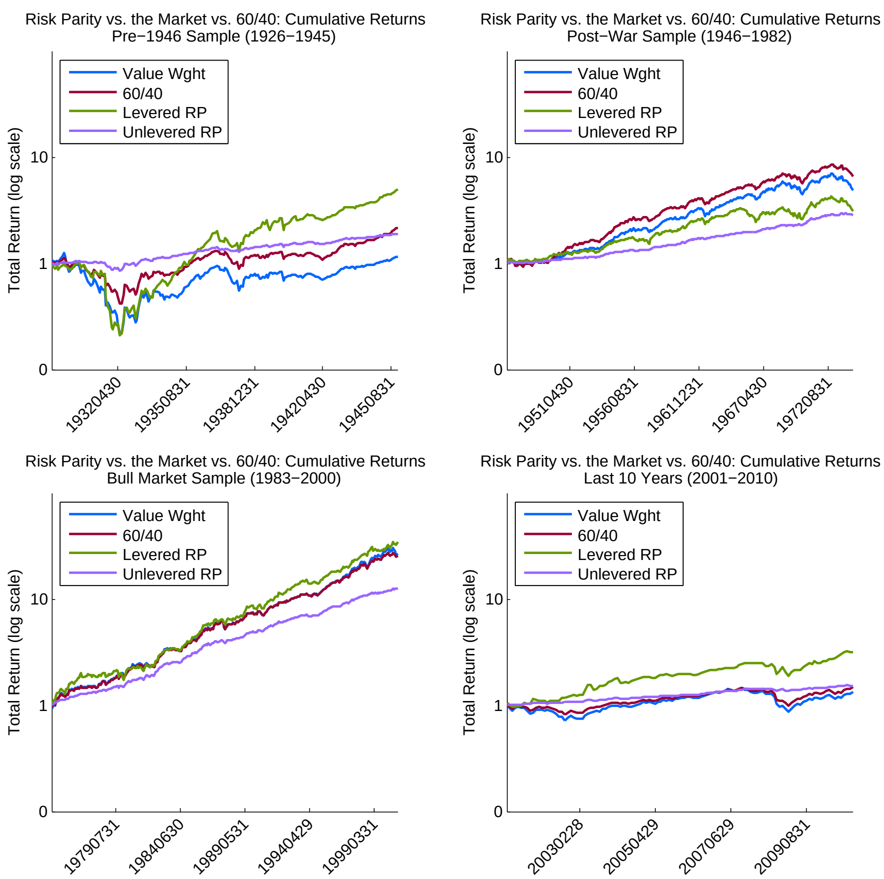
Notes: Performance statistics on the four strategies over the period 1926–2010. In Panel A, the levered risk parity strategy is financed at the 90-Day T-Bill Rate. In Panels B and C, the levered risk parity strategy is financed at the 3-Month Euro-Dollar Deposit Rate. In Panel C, the 60/40 and risk parity strategies are adjusted for turnover.

Figure 3.1: Cumulative Returns to Strategies over the Long Sample



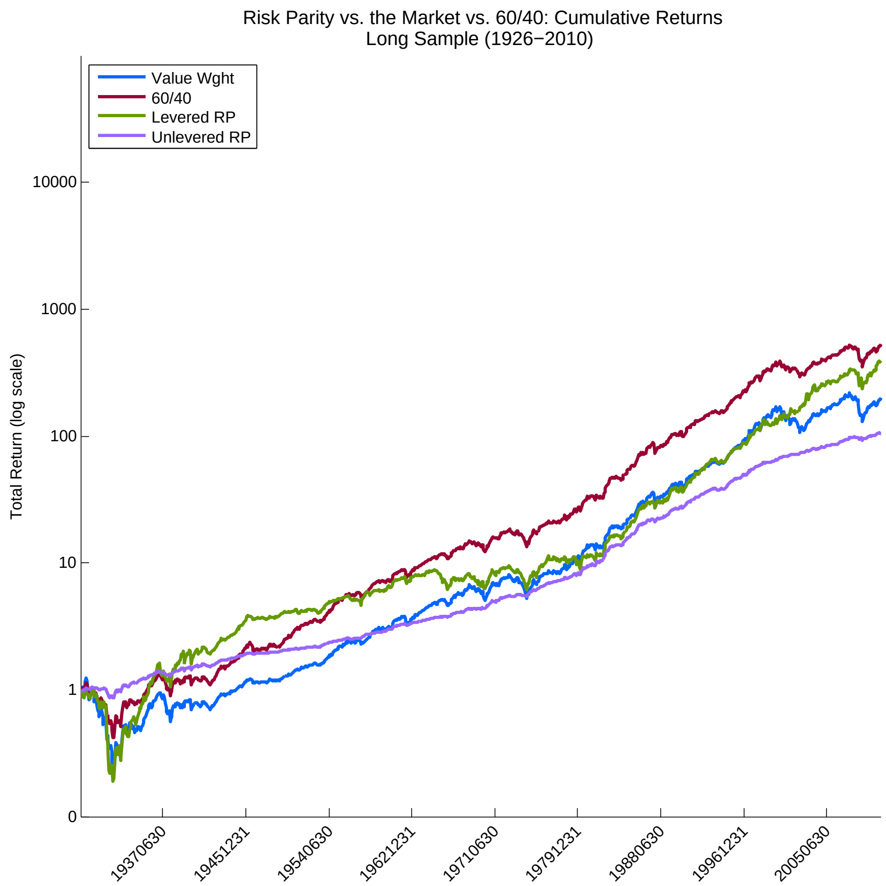
Monthly compounded returns to four strategies based on US Equity and US Treasury Bonds over the period 1926–2010. The levered risk parity strategy was financed at the 90-Day T-Bill Rate.

Figure 3.2: Cumulative Returns to Strategies over Sub-Periods



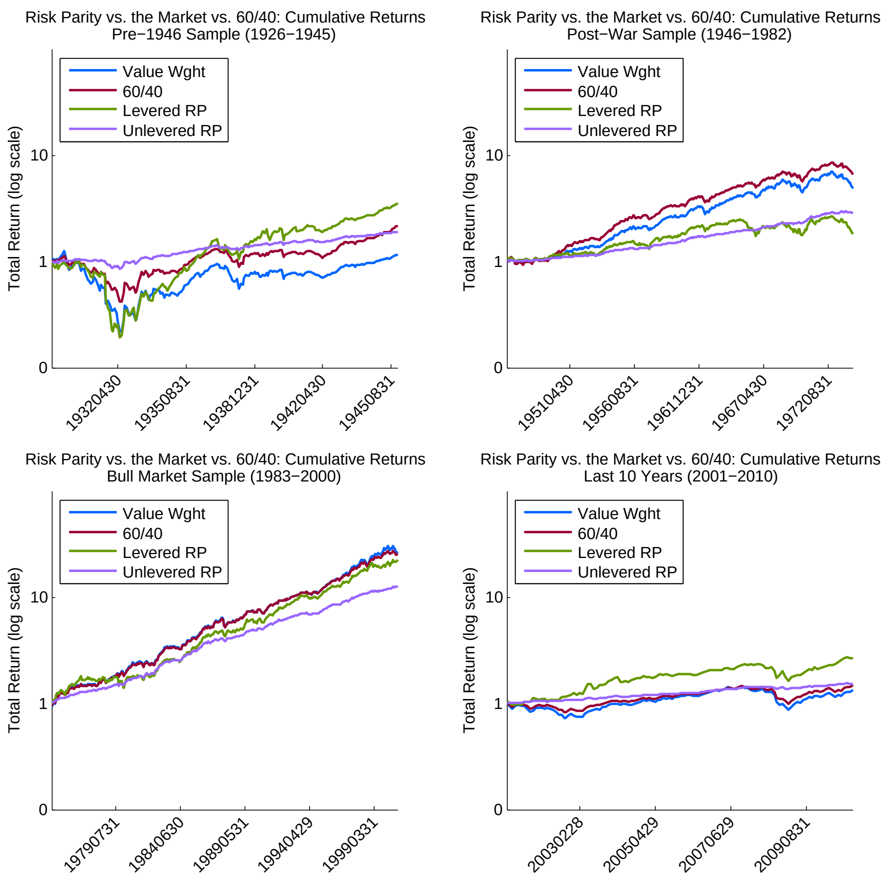
Monthly compounded returns to four strategies based on US Equity and US Treasury Bonds over 4 sub-periods. The levered risk parity strategy was financed at the at the 90-Day T-Bill Rate. The results depend materially on the evaluation period.

Figure 3.3: Cumulative Returns to Strategies over the Long Sample



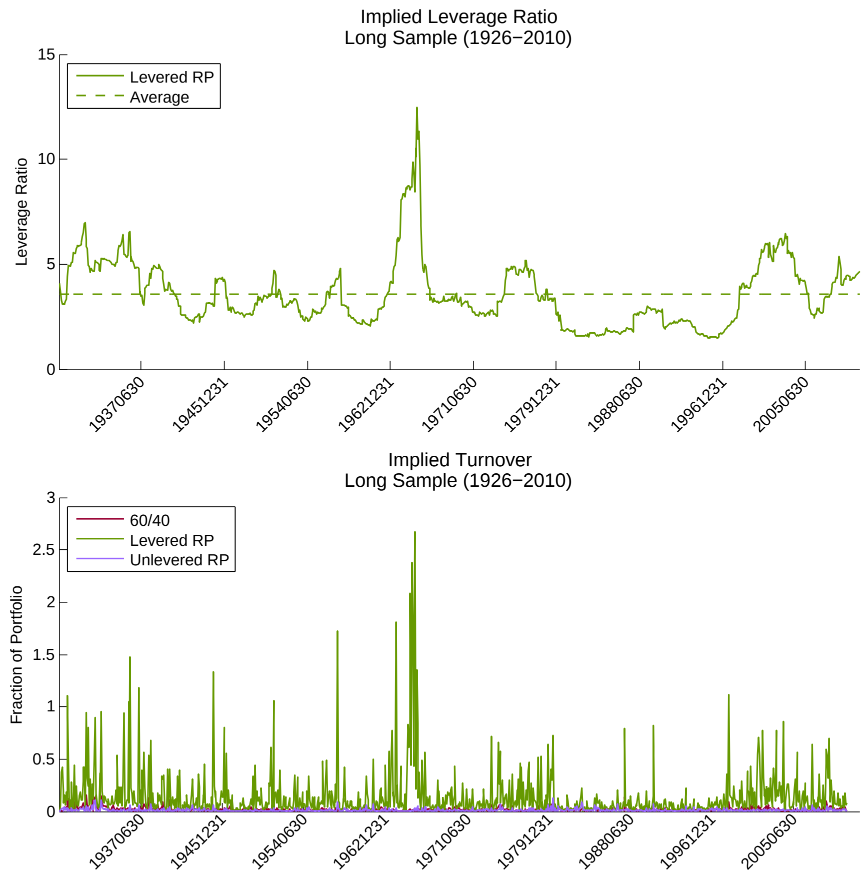
Monthly compounded returns to four strategies based on US Equity and US Treasury Bonds over the period 1926–2010. The levered risk parity strategy was financed at the a 3-Month Euro-Dollar Deposit Rate. A comparison with figure 3.1 shows the magnitude of the performance drag.

Figure 3.4: Cumulative Returns to Strategies over Sub-Periods



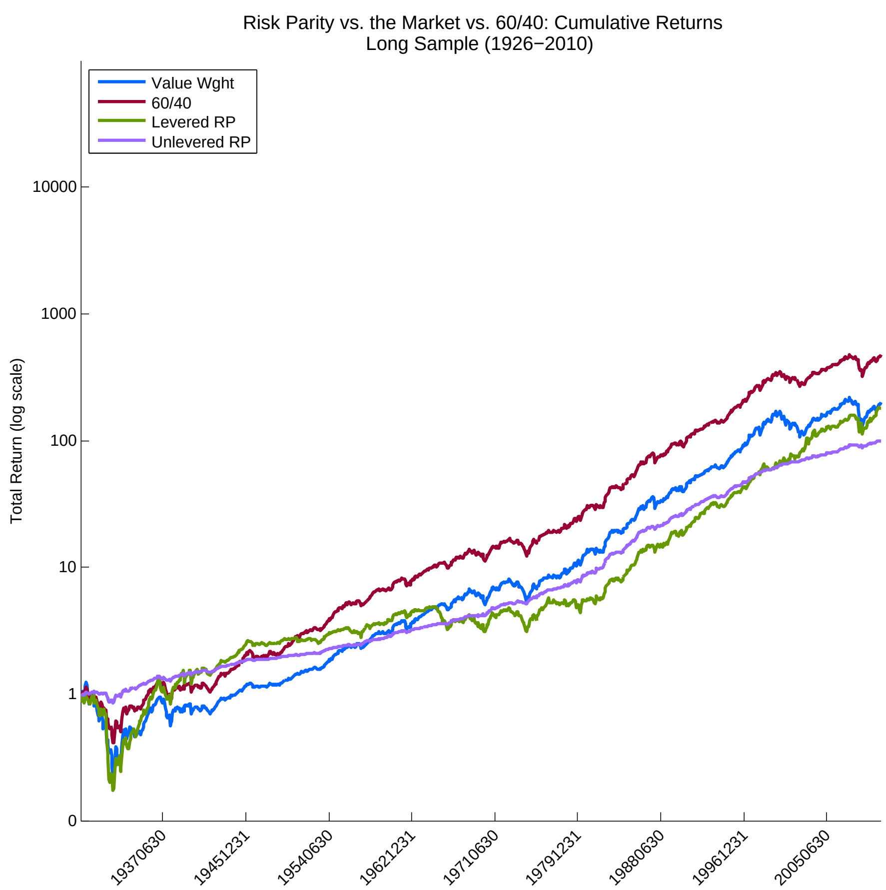
Monthly compounded returns to four strategies based on US Equity and US Treasury Bonds over 4 sub-periods. The levered risk parity strategy was financed at the 3-Month Euro-Dollar Deposit Rate. A comparison with figure 3.2 shows the magnitude of the performance drag, which was most severe in the Post-War sample.

Figure 3.5: Leverage and Turnover



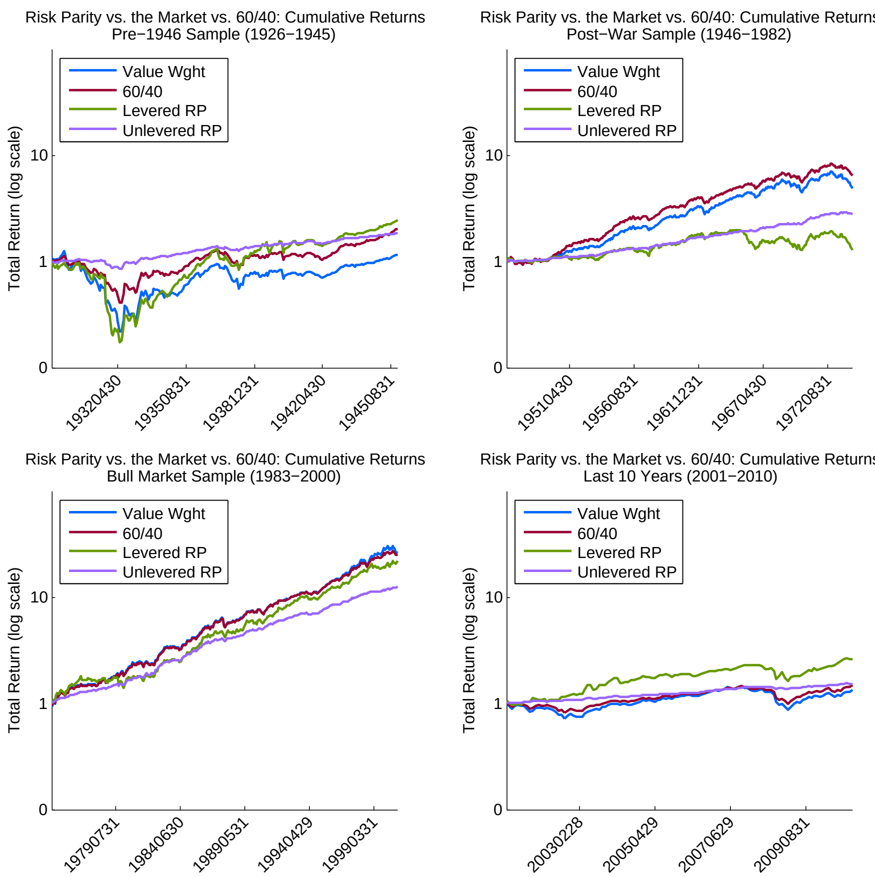
Strategy turnover. The top panel plots the leverage required in order for the estimated volatility of the risk parity strategy to match the estimated volatility of the market at each rebalancing. The average over the entire period was 3.55. The spike in leverage occurred on September 30, 1965, which was a rare moment when bond volatility was relative low (.5%), and both equity volatility (10%) and market weight (72%) were relatively high. The bottom panel shows the turnover of the risk parity and 60/40 strategies at each rebalancing.

Figure 3.6: Cumulative Returns to Strategies over the Long Sample



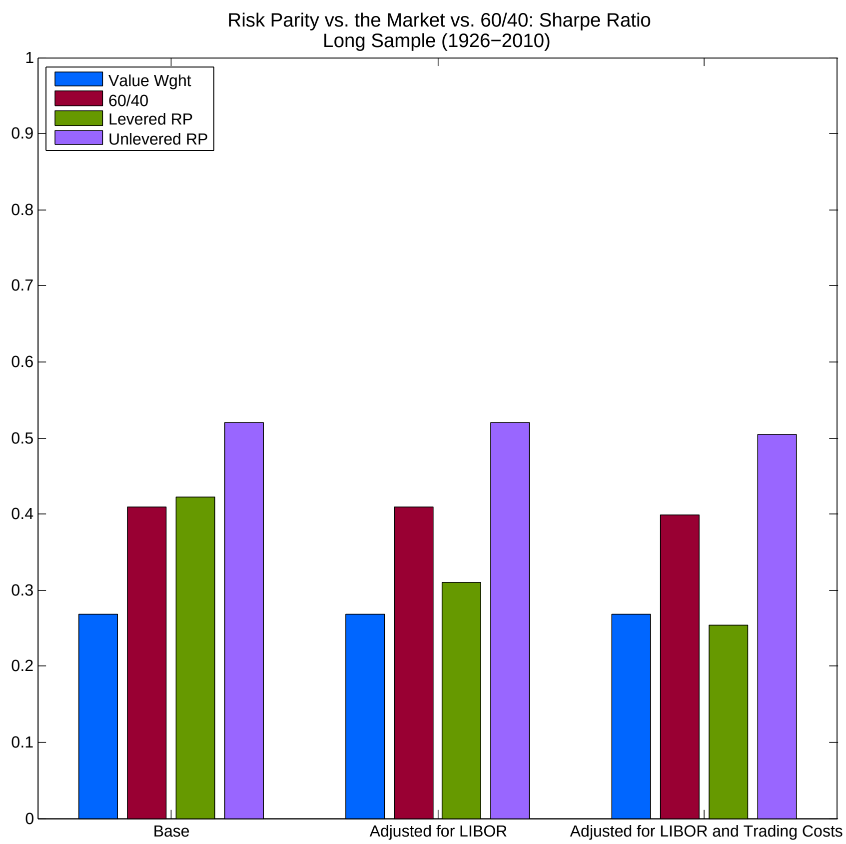
Monthly compounded returns to four strategies based on US Equity and US Treasury Bonds over the period 1926–2010. The levered risk parity strategy was financed at the a 3-Month Euro-Dollar Deposit Rate and adjustments are made for turnover. A comparison with figure 3.3 shows the magnitude of the performance drag.

Figure 3.7: Cumulative Returns to Strategies over Sub-Periods



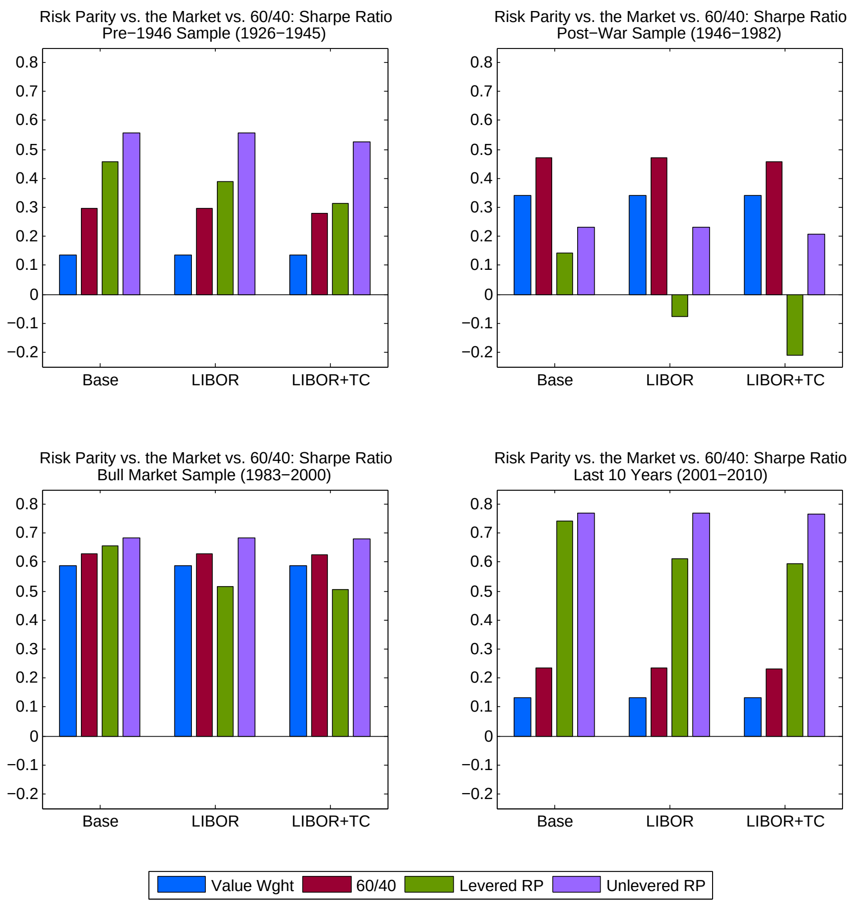
Monthly compounded returns to four strategies based on US Equity and US Treasury Bonds over 4 sub-periods. The levered risk parity strategy was financed at the 3-Month Euro-Dollar Deposit Rate and adjustments are made for turnover. A comparison with figure 3.4 shows the magnitude of the performance drag.

Figure 3.8: Realized Sharpe Ratios over the Long Sample



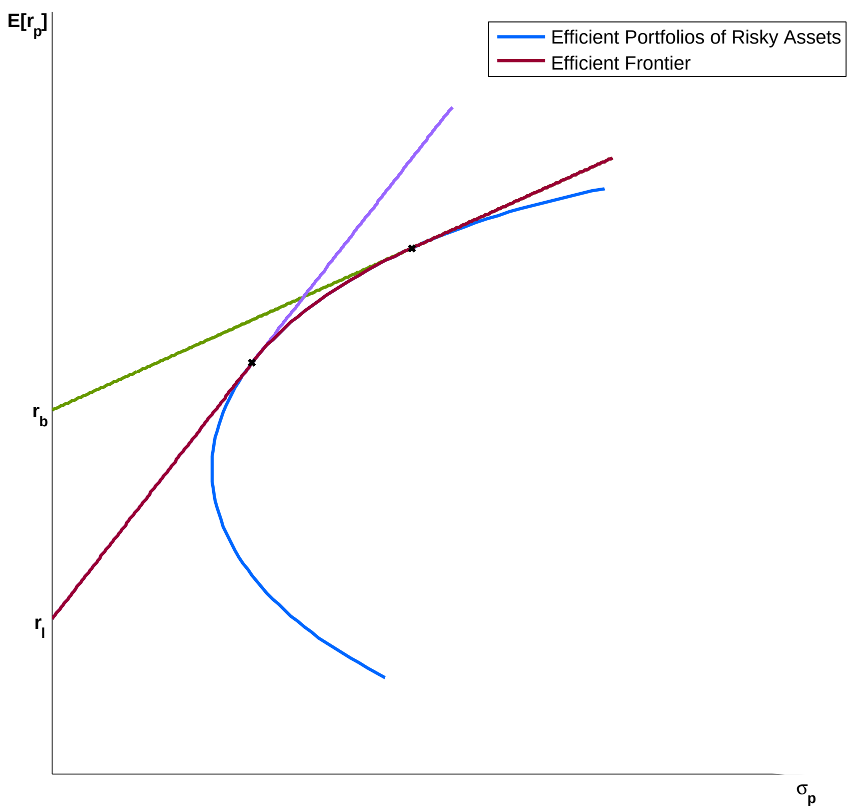
Realized Sharpe ratios for the four strategies over the period 1926–2010. Unlevered risk parity dominates, even before adjustment for market frictions.

Figure 3.9: Realized Sharpe Ratios over Sub-Periods



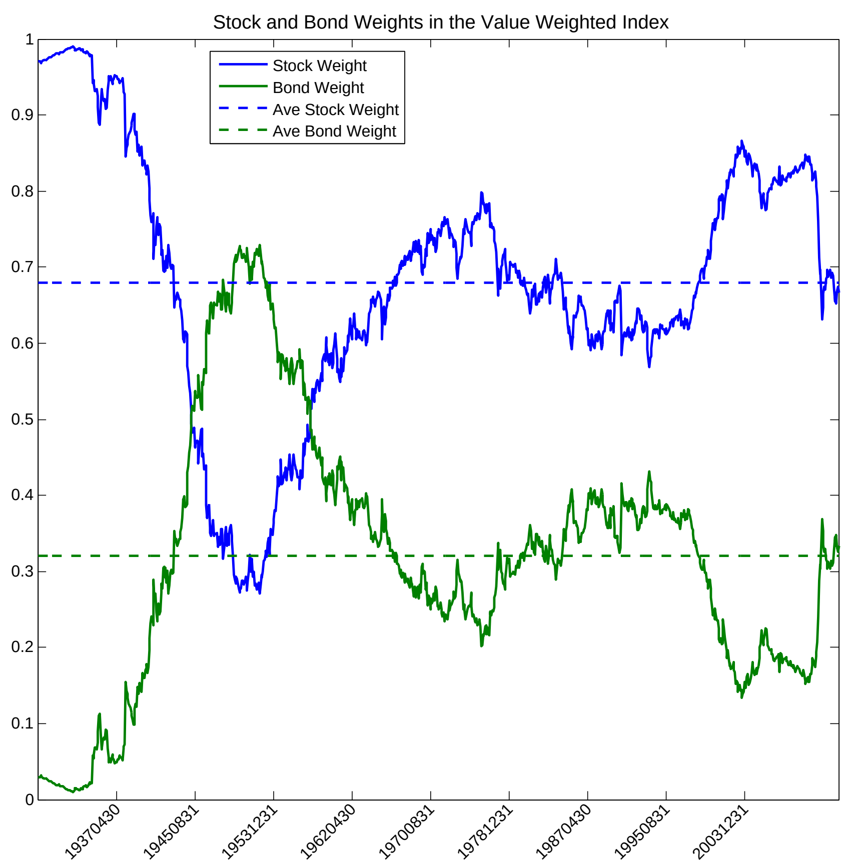
Realized Sharpe ratios for the four strategies over the four sub-periods. Apart from the Post-War Sample, Unlevered risk parity dominates, even before adjustment for market frictions.

Figure 3.10: Capital Market Line



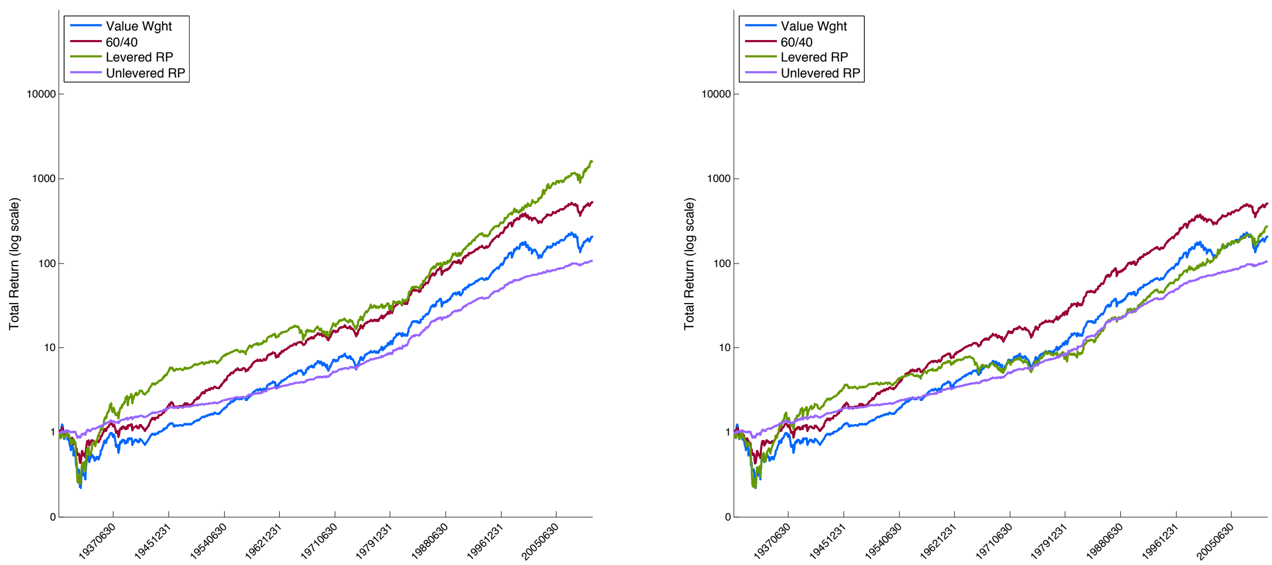
When the rate of borrowing is higher than the risk-free rate, the Capital Market Line in the standard mean-variance diagram has three components. The ex ante Sharpe ratio of a levered portfolio consisting of the market portfolio and cash is lower than the ex ante Sharpe ratio of the market portfolio.

Figure 3.11: Stock and Bond Weights



Weights for stocks and bonds implied by market capitalization over the sample period.

Figure 3.12: Cumulative Returns to Strategies over the Long Sample with Annual Rebalancing



Monthly compounded returns to four strategies based on US Equity and US Treasury Bonds over the period 1926–2010 with annual rebalancing. In the left panel, the levered risk parity strategy was financed at the 90-Day T-Bill Rate and no adjustments are made for turnover. In the right panel, the levered risk parity strategy was financed at the 3-Month Euro-Dollar Deposit Rate and adjustments are made for turnover.

Conclusion

This dissertation examined several aspects of so-called low-risk investing. If investors care about higher moments of return distributions, beyond mean and variance, then low-risk may be a complete misnomer. Chapter 1 showed that despite the fact that low-beta portfolios are generally less volatile than higher beta portfolios, they are not less risky. Low-beta portfolios face higher excess kurtosis than higher beta portfolios. The non-normality of security returns is well recognized and there is much empirical evidence that investors do indeed care about excess kurtosis. However, the connection between low-beta and excess kurtosis has not been previously recognized. In the long-run, this results in higher excess return per unit of volatility, though it does not necessarily result in higher absolute returns.

In order to meet absolute return targets, many investors turn to leverage. But leverage involves its own perils, beyond that of levering up on non-volatility risk. Chapter 2 showed that even gross of "risky" [18] borrowing and trading costs, any strategy that involves dynamic leverage adds considerable noise to returns through a covariance term, i.e., the covariance of leverage and excess borrowing return. Though theory does not rule out the possibility of this covariance being positive, and thus enhancing returns, for some popularly used low-risk strategies over our 84-year sample the covariance was negative and a substantial drag on returns.

Further, inclusion of realistic borrowing and trading costs make leverage, whether constant or dynamic, a tool to be used with caution. Chapter 3 demonstrated this in the dynamic setting with risk parity strategies. There have been some periods, most notably the recent past, where risk parity strategies have outperformed after accounting for all transaction costs (borrowing plus trading). However, there have also been periods longer than the investment horizon of even the most long-term investors, where they have substantially underperformed on a transaction cost adjusted basis.

The bottom line for low-risk investing is, as for most things that look too good to be true, "buyer beware."

[18] From the perspective of the lender, who will charge a higher than risk-free rate to lend to anyone but the government.

Appendix A

The Low-Beta Anomaly

A.1 Data and Processing

All data used in this chapter came from the Kenneth R. French data library (`http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/Data_Library/f-f_factors.html`) and the Center for Research in Security Prices (CRSP) U.S. stock databases. The daily market (which includes all NYSE, AMEX, and NASDAQ stocks) excess returns and the daily 1-month Treasury Bill rate (i.e., the risk-free rate) are taken from the daily Fama-French research factors file. All individual stock betas are estimated with respect to the daily market excess returns in this file. Daily and monthly stock data are taken from the CRSP daily and monthly stock files. Once downloaded, the CRSP data were cleaned and refined according to the following criteria:

- Duplicates were removed, where duplicates were defined as multiple occurrences of *obsdate* and *ticker*.
- Records containing non-numeric values for *prc*, *ret*, and *shrout* were removed.
- For each month in the period January 1990 through December 2011, stocks that matched the following conditions were then selected:
  - The number of observations was greater than 17 (greater than 14 for September 2001).
  - The *shrcd* was in (10,11,12) – these are the codes for ordinary common shares.
  - The *exchcd* was in (10,11,12) – these are the codes for NYSE, AMEX, and NASDAQ.
- Market capitalization was set to $prc \times shrout$ for each record.

For each stock in each month, a beta was estimated based on 6-months of trailing daily data, where the data was required to come from the previous 6 contiguous months. This resulted in beta estimations for stocks in all months from June 1990 through November 2011, a total of 258 months. The average number of stocks processed per month over this period was 5,461, with a maximum of 7,127 in October 1997 and a minimum of 4,022 in November 2011. As described in the text, quantile regressions of stock excess returns on their betas were then run over the entire sample period, where the dependent variables were the monthly excess returns of the stocks in months *following* the 6-month period that was used to estimate their betas. In order to avoid estimation issues with very small stocks, 99% of the largest stocks (by market capitalization) in each month were used for the quantile regressions. This left an average of 3,041 stocks per month, with a maximum of 4,198 in October of 1997, and a minimum of 2,271 in November of 2011. Each quantile regression used a total of 784,658 stock excess returns and stock betas.

A.2 Beta Rank Correlations

Given the facts that new betas were estimated for each stock in each month in the sample and that the 6-month rolling estimation window is relatively short, there may be concern that the time-series of beta estimates for any particular stock are too noisy to say anything useful about low-beta portfolios. If estimated betas changed significantly from month-to-month then the first tercile and first quintile portfolios may essentially be random collections of stocks and their risk-adjusted outperformance could be a fluke. Further, noisier estimates imply higher turnover and higher transaction costs for an investor trying to follow a simple strategy of low-beta investing based on monthly beta sorts. [1] To address this concern, beta rank correlations were calculated and for lags of 1, 6, 12, and 60 months, and for three levels of granularity:

1. Using individual stock betas,
2. Sorting stocks by beta and forming 100 portfolios and assigning each stock in a portfolio the capitalization-weighted portfolio beta,
3. Sorting stocks by beta and forming 10 portfolios and assigning each stock in a portfolio the capitalization-weighted portfolio beta.

The results are plotted in figure 1.14. The differences between the different levels of granularity are minimal. Looking at the results based on individual stocks, at the 1-month lag the beta rank correlations are highly variable, but always stay above 0.50, with an average of 0.80 over the sample period. The variability is somewhat less at longer lags due to the

[1] In principle, the first tercile and first quintile portfolios are investable, since their returns were measured in the month following the period used to estimate the portfolio formation betas.

fact that stocks were only included in the rank correlation calculation if they were present in both the base period and the lagged period. Even at a 60-month lag, there is positive beta rank correlation, with an average of 0.24 over the sample period. Though not a rigorous analysis, these results suggest that noise in the beta estimates and turnover [2] may not be of significant concern.

A.3 Standard Errors for Risk Premium Estimates

Given the linear structural equation

$$R = X\kappa_{\theta,0} + \epsilon, \tag{A.1}$$

under classical assumptions (see Powell (1994)), the asymptotic distribution of the quantile regression estimator is given by

$$\sqrt{N}(\hat{\kappa}_\theta - \kappa_{\theta,0}) \xrightarrow{d} \mathcal{N}\left(0, \theta(1-\theta)H^{-1}VH^{-1}\right), \tag{A.2}$$

where [3]

$$V = \underset{N\to\infty}{\text{plim}}\hat{V}, \quad \hat{V} = \frac{1}{N}X'X, \quad H = \underset{N\to\infty}{\text{plim}}\hat{H}, \quad \hat{H} = \frac{1}{N}X'\hat{F}X, \tag{A.3}$$

$\hat{F} = \mathbf{D}iag(\hat{f}(\epsilon|X))$ is an estimate of the conditional density of the error terms, and N is the number of observations (stock excess returns). Given a weighting matrix W, the form of the asymptotic distribution of the quantile regression estimator is the same with

$$\hat{V} = \frac{1}{N}X'WX, \quad \hat{H} = \frac{1}{N}X'\hat{F}WX. \tag{A.4}$$

In this chapter, X has dimension $N \times 259$ (i.e., one column for each month with ones indicating observations from that month and zeros otherwise, and one column of estimated betas), $\kappa = (\alpha, \lambda)$ has dimension 259×1, $\hat{F}$ has dimension $N \times N$, and $W = \mathbf{D}iag(w)$ has dimension $N \times N$, where w is the length N vector of stock capitalization weights. Letting $\hat{C} = \theta(1-\theta)\hat{H}^{-1}\hat{V}\hat{H}^{-1}$, which has dimension 259×259, the estimate of the standard error of $\hat{\lambda}_\theta$ is given by the square root of $\hat{C}(259, 259)/N$.

However, there are several departures from the classical assumptions that need to be accounted for in this application:

- The error terms are cross-sectionally heteroscedastic. If they were not, then the risk premium estimates would be the same for all quantiles (i.e., the quantile lines would be parallel).

[2] This chapter has not modeled transaction costs, which could erase the apparent risk-adjusted outperformance of low-beta portfolios in the face of high turnover.

[3] The operator *plim* stands for *limit in probability*.

- The error terms are serially correlated.
- The independent variable (beta) in the quantile regressions is *estimated*.

To address heteroscedasticity and serial correlation in the error terms, a matrix ψ is formed where each row is a time average of the rows corresponding to each unique stock in the sample

$$\psi_{i\cdot} = \frac{1}{T_i - t_i + 1} \sum_{j=t_i}^{T_i} X_{i,j}(\theta - \mathbf{1}[\epsilon_{i,j} < 0]), \tag{A.5}$$

where t_i is the first month for which stock i is in the sample, T_i is the last month for which stock i is in the sample, $X_{i,j}$ (dimension 1×259) is the row of X corresponding to stock i in month j, $\epsilon_{i,j}$ is the scalar error term corresponding to stock i in month j, and $(\theta - \mathbf{1}[\epsilon_{i,j} < 0])$ is the derivative of the so-called check function (see Koenker (2005)) $\rho_\theta(\epsilon) = \epsilon(\theta - \mathbf{1}[\epsilon < 0])$. Then

$$\psi = (\psi_{1\cdot}', \ldots, \psi_{M\cdot}')', \tag{A.6}$$

where M is the number of unique stocks in the sample (ψ has dimension $M \times 259$), and

$$V_\psi = \operatorname*{plim}_{M\to\infty} \hat{V}_\psi, \quad \hat{V}_\psi = \frac{1}{M}\hat{\psi}'\bar{W}\hat{\psi}, \tag{A.7}$$

where $\bar{W} = \mathbf{D}iag(\bar{w})$ and $\bar{w}$ is the normalized sum (over time) of the capitalization-weights for each unique stock. In the presence of heteroscedasticity and serial correlation (but non-estimated regressors), and under the assumption that the error terms are uncorrelated across stocks, [4] the asymptotic distribution of the quantile regression estimator is then given by

$$\sqrt{N}(\hat{\kappa}_\theta - \kappa_{\theta,0}) \xrightarrow{d} \mathcal{N}\left(0, H^{-1}V_\psi H^{-1}\right). \tag{A.8}$$

To correct for estimated regressors, an estimate of the asymptotic covariance matrix of the betas from the first stage regressions is given by

$$J = \operatorname*{plim}_{N\to\infty} \hat{J}, \quad \hat{J} = \frac{1}{N}X'W\hat{Z}\hat{D}\hat{Z}'\hat{F}X, \tag{A.9}$$

where $\hat{Z}$ is an $N \times 2N$ block diagonal matrix with each block being a row vector of length 2, with a one in the first position and the mean of the excess market returns used for estimating that observation's beta in the second position, and $\hat{D}$ is an $2N \times 2N$ block diagonal matrix with each block being a 2×2 matrix of the estimated covariance matrix for that observation's estimated alpha and beta. Hence, J has dimension 259×259. Then with all corrections, the asymptotic distribution of the quantile regression estimator is given by

$$\sqrt{N}(\hat{\kappa}_\theta - \kappa_{\theta,0}) \xrightarrow{d} \mathcal{N}\left(0, H^{-1}(V_\psi + J)H^{-1}\right). \tag{A.10}$$

[4] Though known to not strictly hold, this assumption is common in financial applications.

Table A.10: Risk Premium 99% Confidence Intervals for $\theta \in \Theta$

θ	CI($\hat{\lambda}_\theta$)	θ	CI($\hat{\lambda}_\theta$)	θ	CI($\hat{\lambda}_\theta$)	θ	CI($\hat{\lambda}_\theta$)	θ	CI($\hat{\lambda}_\theta$)
0.5	(-7.18,-7.05)	21.0	(-2.37,-2.33)	42.0	(-0.74,-0.72)	63.0	(0.62,0.64)	84.0	(2.43,2.46)
1.0	(-6.25,-6.15)	22.0	(-2.27,-2.24)	43.0	(-0.68,-0.65)	64.0	(0.69,0.71)	85.0	(2.55,2.59)
2.0	(-5.39,-5.31)	23.0	(-2.18,-2.15)	44.0	(-0.60,-0.58)	65.0	(0.76,0.78)	86.0	(2.64,2.68)
3.0	(-4.86,-4.80)	24.0	(-2.11,-2.07)	45.0	(-0.54,-0.51)	66.0	(0.81,0.84)	87.0	(2.77,2.81)
4.0	(-4.50,-4.44)	25.0	(-2.03,-1.99)	46.0	(-0.46,-0.44)	67.0	(0.89,0.91)	88.0	(2.90,2.94)
5.0	(-4.26,-4.20)	26.0	(-1.95,-1.92)	47.0	(-0.40,-0.38)	68.0	(0.98,1.00)	89.0	(3.06,3.10)
6.0	(-4.05,-4.00)	27.0	(-1.91,-1.88)	48.0	(-0.35,-0.33)	69.0	(1.06,1.09)	90.0	(3.18,3.22)
7.0	(-3.89,-3.84)	28.0	(-1.84,-1.81)	49.0	(-0.30,-0.28)	70.0	(1.16,1.18)	91.0	(3.33,3.37)
8.0	(-3.76,-3.71)	29.0	(-1.75,-1.72)	50.0	(-0.24,-0.22)	71.0	(1.24,1.26)	92.0	(3.53,3.57)
9.0	(-3.66,-3.61)	30.0	(-1.67,-1.64)	51.0	(-0.18,-0.16)	72.0	(1.33,1.35)	93.0	(3.76,3.81)
10.0	(-3.50,-3.45)	31.0	(-1.58,-1.56)	52.0	(-0.12,-0.10)	73.0	(1.42,1.45)	94.0	(4.01,4.06)
11.0	(-3.38,-3.33)	32.0	(-1.49,-1.47)	53.0	(-0.07,-0.05)	74.0	(1.50,1.53)	95.0	(4.23,4.28)
12.0	(-3.25,-3.20)	33.0	(-1.42,-1.39)	54.0	(-0.01,0.01)	75.0	(1.57,1.60)	96.0	(4.56,4.62)
13.0	(-3.17,-3.13)	34.0	(-1.35,-1.32)	55.0	(0.07,0.09)	76.0	(1.64,1.67)	97.0	(4.90,4.96)
14.0	(-3.06,-3.02)	35.0	(-1.28,-1.25)	56.0	(0.12,0.15)	77.0	(1.71,1.74)	98.0	(5.43,5.51)
15.0	(-2.95,-2.91)	36.0	(-1.20,-1.17)	57.0	(0.19,0.21)	78.0	(1.81,1.84)	99.0	(6.36,6.45)
16.0	(-2.87,-2.83)	37.0	(-1.12,-1.10)	58.0	(0.26,0.28)	79.0	(1.88,1.90)	99.5	(7.14,7.26)
17.0	(-2.75,-2.72)	38.0	(-1.06,-1.03)	59.0	(0.33,0.35)	80.0	(1.97,2.00)		
18.0	(-2.66,-2.62)	39.0	(-0.98,-0.96)	60.0	(0.38,0.40)	81.0	(2.08,2.11)		
19.0	(-2.56,-2.52)	40.0	(-0.91,-0.88)	61.0	(0.45,0.47)	82.0	(2.19,2.22)		
20.0	(-2.46,-2.42)	41.0	(-0.82,-0.80)	62.0	(0.53,0.55)	83.0	(2.29,2.32)		

Notes: The confidence intervals in the table are in percent per *month*.

Letting $\hat{C}_{\psi,J} = \hat{H}^{-1}(\hat{V}_\psi + \hat{J})\hat{H}^{-1}$, the estimate of the standard error of $\hat{\lambda}_\theta$ is given by the square root of $\hat{C}_{\psi,J}(259, 259)/N$.

To estimate F we followed Powell (1991) and let

$$\hat{f}(\epsilon_i|X_i) = \frac{1}{2h}\mathbf{1}(|R_i - X_i'\hat{\kappa}_\theta| < h), \tag{A.11}$$

where $h \to 0$ and $\sqrt{N}h \to \infty$ as $N \to \infty$. For this application h was set equal to $N^{-1/3}$.

Table A.10 displays the 99% confidence intervals for all $\hat{\lambda}_\theta$.

A.4 Risk Premium Estimates and Simulation Results Based on LAD Betas

Martin and Timin (1999) argue that betas estimated via OLS are often distorted in the presence of outliers, if the outliers are generated by heavy-tailed distributions; and there is a fair amount of empirical evidence that stock returns contain outliers that come from heavy-tailed distributions. They contend that if this is the case, then the OLS estimate of beta is no longer BLUE (the best linear unbiased estimate). To overcome this issue, Martin and Timin suggest estimating beta with an estimator that is *robust*, in the sense that it is less influenced by outliers, but performs nearly as well as OLS in the absence of outliers. As a robustness check of the results in this chapter, all stock betas were re-estimated using least absolute deviations (LAD) regression (i.e., median regression). All of the quantile regressions were then re-run to estimate risk premia conditioned on LAD betas, the results are plotted in figure 1.13. While the individual stock betas can be quite different based on the OLS or LAD estimate, overall the estimated risk premia are strikingly close. The simulations based on changes in beta implied by the first tercile and first quintile portfolios were also re-run, where the capitalization-weighted OLS betas were used to determine the monthly changes in beta, but the re-estimated risk premia were used as the derivatives with respect to beta. The simulation results are presented in table A.11. The overall results of the chapter are robust to risk premia estimation conditioned on LAD betas.

A.5 A Simple Implementation of Quantile Regression Based on Duality

A.5.1 Risk Identifiers and Risk Envelopes

[5] A functional $\mathcal{D} : \mathbf{L}^2(\Omega) \to [0, \infty)$ is a regular deviation measure if and only if it has a representation of the form

$$\mathcal{D}(X) = \mathbb{E}(X) - \inf_{\mathcal{Q} \in \mathcal{Q}^*} \mathbb{E}(X\mathcal{Q}),$$

for $\mathcal{Q}^* \subset \mathbf{L}^2(\Omega)$, where

(Q1) $\mathcal{Q}^*$ is non-empty, closed, and convex,

(Q2) for every non-constant $X \in \mathbf{L}^2(\Omega)$, $\exists \mathcal{Q} \in \mathcal{Q}^* : \mathbb{E}(X\mathcal{Q}) < \mathbb{E}(X)$,

(Q3) $\mathbb{E}(\mathcal{Q}) = 1, \forall \mathcal{Q} \in \mathcal{Q}^*$.

$\mathcal{Q}^*$ is called the *risk envelope* corresponding to $\mathcal{D}$ and elements of the set $\{\mathcal{Q} \in \mathcal{Q}^* | \mathcal{D}(X) = \mathbb{E}(X) - \mathbb{E}(X\mathcal{Q})\}$ are called the *risk identifiers* for X with respect to $\mathcal{D}$. If the following property is also satisfied

(Q4) $\mathcal{Q} \geq 0, \forall \mathcal{Q} \in \mathcal{Q}^*$,

then each $\mathcal{Q} \in \mathcal{Q}^*$ may be regarded as the density relative to $\mathbb{P}$ of some probability measure $\mathbb{P}'$ on Ω:

$$\mathcal{Q} = d\mathbb{P}'/d\mathbb{P}, \quad \mathbb{P}' = \mathcal{Q}\mathbb{P}.$$

[5] Rockafellar et al (2006).

Hence, the difference

$$\mathbb{E}(X) - \mathbb{E}(X\mathcal{Q}) = \mathbb{E}_{\mathbb{P}}(X) - \mathbb{E}_{\mathbb{P}'}(X).$$

assesses how much "worse" the expectation of X is under $\mathbb{P}'$ versus $\mathbb{P}$. Note that

$$\mathbb{E}(X) - \mathbb{E}(X\mathcal{Q}) = \mathbb{E}[(\mathbb{E}(X) - X)\mathcal{Q}] = \mathbf{C}ov(\mathbb{E}(X) - X, \mathcal{Q}) = \mathbf{C}ov(-X, \mathcal{Q}),$$

thus the risk identifiers for X are the elements of $\mathcal{Q}^*$ that track the "downside" of X as closely as possible.

A.5.2 $\mathbf{C}VaR$ Linear Regression as a Linear Programming Problem

[6] Suppose X and Y are discrete random variables with probability distribution

$$\mathbb{P}[X = x_k, Y = y_k] = p_k, \quad \text{for } k = 1, \ldots, K,$$

where $K = \dim(\Omega)$, and the following linear structural equation is assumed to hold

$$Y = \alpha + X\eta + \epsilon,$$

where $\alpha, \eta \in \mathbb{R}$. The analyst would like to carry out the regression with respect to $\mathbf{C}VaR$-deviation at confidence level $\theta \in (0,1)$. Thus, the following problem needs to be solved

$$\min_{\eta} \ \mathbf{C}VaR^{\Delta}_{\theta}(Y - X\eta),$$

where for some random variable Z, $\mathbf{C}VaR^{\Delta}_{\theta}(Z) \equiv \mathbf{C}VaR_{\theta}(Z - \mathbb{E}[Z])$. The minimizer is the conditional quantile function of Y, $\mathbb{Q}_Y(\theta|X)$. Given $\hat{\eta}$, the intercept is set as $\hat{\alpha} = \mathbb{Q}_{Y-X\hat{\eta}}(\theta)$. Though the objective function is convex and amenable to minimization, $\hat{\eta}$ can also be estimated using an approach based on duality which involves solving the linear programming problem

$$\begin{aligned}
\max_{\mathbf{q}} \quad & \sum_{k=1}^{K} p_k q_k(\mathbb{E}[Y] - y_k) \\
\text{s.t.} \quad & \sum_{k=1}^{K} p_k q_k = 1, \\
& \sum_{k=1}^{K} p_k q_k(\mathbb{E}[X] - x_k) = 0, \\
& 0 \le q_k \le 1/\theta, \quad \text{for } k = 1, \ldots, K,
\end{aligned}$$

where $\mathbf{q} = (q_1, \ldots, q_K)'$. Then $\hat{\eta}$ is the Lagrange multiplier of the second constraint. Note that $\mathcal{D}(Z) = \mathbf{C}VaR^{\Delta}_{\theta}(Z)$ corresponds to $\mathcal{Q}^* = \{\mathcal{Q} | 0 \le \mathcal{Q} \le 1/\theta, \mathbb{E}(\mathcal{Q}) = 1\}$.

[6] Rockafellar et al (2002).

Table A.11: Cross-Sectional Simulation Statistics (LAD Betas)

First Tercile Betas	Estimated RP	Market RP	Difference	t-Statistic
Average Excess Return	7.33	5.96	1.37	14.91
Excess Return Volatility	23.01	28.78	-5.77	-14.95
Sharpe Ratio	0.40	0.26	0.14	3.65
Skewness	6.72	6.45	0.27	0.35
Excess Kurtosis	505.89	332.97	172.92	9.03
Downside Volatility	15.18	18.49	-3.31	-14.95
Expected Shortfall (95%)	14.11	17.63	-3.52	-14.81

First Tercile Betas	Estimated RP	De-Levered RP	Difference	t-Statistic
Average Excess Return	7.33	2.94	4.39	1.99
Excess Return Volatility	23.01	11.59	11.42	8.71
Sharpe Ratio	0.40	0.26	0.14	3.60
Skewness	6.72	5.95	0.77	0.86
Excess Kurtosis	505.89	332.94	172.94	8.83
Downside Volatility	15.18	7.43	7.76	9.24
Expected Shortfall (95%)	14.11	7.05	7.07	8.11

First Quintile Betas	Estimated RP	Market RP	Difference	t-Statistic
Average Excess Return	7.55	6.00	1.55	17.21
Excess Return Volatility	22.27	28.79	-6.52	-18.74
Sharpe Ratio	0.43	0.26	0.17	3.98
Skewness	7.06	6.44	0.62	0.71
Excess Kurtosis	531.53	332.62	198.92	10.09
Downside Volatility	14.74	18.50	-3.76	-18.51
Expected Shortfall (95%)	13.65	17.63	-3.98	-18.47

First Quintile Betas	Estimated RP	De-Levered RP	Difference	t-Statistic
Average Excess Return	7.55	1.92	5.63	2.11
Excess Return Volatility	22.27	8.28	13.99	10.22
Sharpe Ratio	0.43	0.26	0.16	1.90
Skewness	7.06	5.83	1.23	0.68
Excess Kurtosis	531.53	333.68	197.85	10.23
Downside Volatility	14.74	5.31	9.43	10.79
Expected Shortfall (95%)	13.65	5.05	8.60	9.38

Notes: In each month, 25,000 samples are drawn from the CDF's defined by the empirical quantiles (Market RP), the quantiles adjusted by the effective risk premiums (De-Levered RP), and the estimated risk premiums (Estimated RP, i.e., pure stock low-beta). The latter two adjustments are based on the change in beta implied by the first tercile portfolio OLS betas (an average change in beta of -0.56) and first quintile portfolio OLS betas (an average change in beta of -0.71) in each month. Expected shortfall (ES) for a random variable X is defined as $\mathrm{ES}_\alpha(X) \equiv -\mathbb{E}[X|X \leq \mathbb{Q}_X(1-\alpha)]$. The reported numbers are the time-series means of the cross-sectional statistics. The t-Statistics are Newey-West t-Statistics for the monthly differences in the cross-sectional statistics.

Appendix B

Levered Portfolios

B.1 Related Literature

B.1.1 CAPM

Finance continues to draw heavily on the Capital Asset Pricing Model (CAPM) developed in Treynor (1962), Treynor and Black (1976), Sharpe (1964), Lintner (1965a), Lintner (1965b), Mossin (1966), and extended in Black and Litterman (1992).[1] In the CAPM, leverage is a means to adjust the level of risk in an efficient portfolio and nothing more. In contrast, Markowitz (2005) illustrated another facet of leverage in the context of a market composed of three coconut farms. In this disarmingly simple example, some investors were leverage-constrained and others were not. The market portfolio was mean-variance inefficient; as a result, no mean-variance investor would choose to hold it, and expected returns of assets did not depend linearly on market betas.

B.1.2 Measurement of Risk and Nonlinearities

An impediment to a clear understanding of leverage may be the way we measure its risk. Standard risk measures such as volatility, value at risk, expected shortfall, and beta scale linearly with leverage. But as we know from the collapse of Long Term Capital, the relationship between risk and leverage can be non-linear; see, for example, Jorion (2000). Föllmer and Schied (2002) and Föllmer and Schied (2011, Chapter 4) described risk measures that penalize leverage in a super-linear way. Recent experience suggests that these measures may be useful in assessing the risk of levered strategies.

One contribution of this chapter is to explain how the interaction between leverage and market frictions creates specific nonlinearities in the relationship between leverage and re-

[1] A history of the CAPM elucidating Jack Treynor's role in its development is in French (2003).

turn. Understanding these specific nonlinearities provides a practical framework to guide the decision on whether and how to lever.

B.1.3 Motivations for Leverage

If investors are overconfident in their predictions of investment returns, they may find leverage attractive because it magnifies the returns when times are good, and because they underestimate the risk of bad outcomes.[2]

Perfectly rational investors may also be attracted to leverage by the low-risk anomaly, the apparent tendency of certain low-risk portfolios to have higher risk-adjusted return than high-risk portfolios. An investor who believes in the low-risk anomaly will be tempted to lever low-risk portfolios, in the hope of achieving high expected returns at acceptable levels of risk.

In a CAPM world, investors with below-average risk aversion will choose to lever the market portfolio.[3] The low-risk anomaly provides a rational argument for investors with typical risk aversion to use leverage. Indeed, the low-risk anomaly is arguably the only rational argument for an investor to use leverage in an investment portfolio composed of publicly traded securities.[4] Differences in risk aversion could explain some investors choosing higher expected return at the price of higher volatility, but there is little reason for a rational investor to choose *leverage* unless the source portfolio being levered offers superior *risk-adjusted* returns, at a volatility *below* the investor's risk tolerance.

B.1.4 Levered Low-Risk Strategies

An early reference to low-risk investing is Markowitz (1952) who commented that a minimum-variance portfolio is mean-variance optimal if all assets returns are uncorrelated and have equal expectations. But low-risk strategies typically require leverage in order to meet expected return targets. In an exploration of this idea, Frazzini and Pedersen (2011) echoed some of the conclusions in Markowitz (2005), and they complemented theory with an empirical study of an implicitly levered equity risk factor that was long low-beta stocks and

[2]A positive relationship between overconfident CEOs and firm leverage is documented in Malmendier et al. (2011). Shefrin and Statman (2011) identified excessive leverage taken by overconfident bankers as a contributor to the global financial crisis.

[3]Note, however, that the market portfolio in CAPM includes bonds and other risky asset classes, rather than just stocks. Levered strategies include the use of margin, and futures and other derivatives, to assemble levered equity-only portfolios, which behave quite differently from levered portfolios in CAPM.

[4]There are, of course, other rational arguments for using leverage in other contexts. The leverage provided by a mortgage may be the only feasible way for a household to buy a house, which provides a stream of consumption benefits and tax advantages in addition to facilitating an investment in the real estate market. Companies leverage their shareholder equity with borrowing to finance operations, for a variety of reasons, including differences in risk aversion, informational asymmetries, and tax implications.

short high-beta stocks. This factor descended from Black, Jensen and Scholes (1972), which provided evidence that the CAPM may not properly reflect market behavior.

B.1.5 Empirical Evidence on Levered Low-Risk Investing

There is a growing empirical literature indicating that market frictions may prevent investors from harvesting the returns promised by a frictionless analysis of levered low-risk strategies. Chapter 3 shows that financing and trading costs can negate the abnormal profits earned by a levered risk parity strategy in a friction-free market. Li et al. (2014) and Fu (2009) showed that market frictions may impede the ability to scale up the return of low-risk strategies through leverage.[5]

Asset allocation that is based on capital weights has a long and distinguished history; see, for example Graham (1949) and Bogle (2007). However, rules-based strategies that allocate risk instead of, or in addition to, capital are of a more recent vintage. Risk-based investing is discussed in chapter 3, Lörtscher (1990), Kessler and Schwarz (1996), Qian (2005), Clarke, de Silva and Thorley (2011), Shah (2011), Sefton, Jessop, Rossi, Jones and Zhang (2011), Clarke et al. (2013), Cowan and Wilderman (2011), Bailey and de Prado (2012), Goldberg and Mahmoud (2013) and elsewhere. Strategies that target volatility are also gaining acceptance, although the literature is still sparse. Goldsticker (2012) compared volatility targeting strategies to standard allocations such as fixed-mix, and found that the relative performance of the strategies was period dependent.

B.1.6 The Effect of Leverage on Markets

Another important question is the extent to which leverage may *contribute* to market instability. See, for example, Brunnermeier and Pedersen (2009), Adrian and Shin (2010) and Geanakoplos (2010). We do not address that question here, as we restrict our analysis to the effect of leverage on the return of investment strategies, taking the distribution of the underlying asset returns as given.

B.1.7 Arithmetic versus Geometric Return

Despite the large literature on the importance of compounding to investment outcomes, analyses of investment strategies are often based on arithmetic expected return. Background references on compounding and geometric return include Fernholz (2002) and MacLean et al. (2011). Perold and Sharpe (1988) discussed how the interplay among volatility, rebalancing and compound return causes a fully-invested fixed-mix or portfolio-insurance strategy to behave differently from a buy-and-hold strategy with the same initial mix. Booth and Fama

[5]Ross (2004) provided an example of the limits to arbitraging mispricings of interest-only strips of mortgage backed securities.

(1992) worked out the relationship between the compound return to a fixed-mix portfolio and its constituents, and their results were applied to portfolios that include commodities in Willenbrock (2011). Markowitz (2012) compared six different mean-variance approximations to geometric return.

B.2 Data

The results presented in this chapter were based on CRSP stock and bond data from January of 1929 through December of 2012. The aggregate stock return is the CRSP value weighted market return (including dividends) from the table *Monthly Stock–Market Indices (NYSE/AMEX/NASDAQ)* – variable name *vwretd*. The aggregate bond return was the face value outstanding (cross-sectionally) weighted average of the unadjusted return for each bond in the *CRSP Monthly Treasury (Master)* table. In this table, the variable name for the unadjusted return is *retnua* and for the face value outstanding is *iout1r*. All bonds in the table were used, provided the values for both *retnua* and *iout1r* were not missing.

The proxy for the risk-free rate was the *USA Government 90-day T-Bills Secondary Market* rate, provided by Global Financial Data (`http://www.globalfinancialdata.com`), covering the period from January of 1929 through December of 2012. The proxy for the cost of financing leverage was the *U.S. 3-Month Euro-Dollar Deposit* rate, downloaded from the Federal Reserve (`http://www.federalreserve.gov/releases/h15/data.htm`). The 3-Month Euro-Dollar Deposit data is available from January of 1971 through December of 2012. Prior to January of 1971, a constant of 60 basis points was added to the 90-day T-Bill rate.[6] Trading costs were calculated using the procedure described in Appendix B.3. We assumed the cost of trading was 100 basis points from 1926 to 1955, 50 basis points from 1956 to 1970, and 10 basis points from 1971 onward.

The construction of the unlevered and levered risk parity strategies was exactly as detailed in chapter 3. The construction of the bonds levered to stock strategies was the analogue for the case of a single asset class.

Following Asness et al. (2012), chapter 3 used the volatility of the market as the target for risk parity. In this chapter, we used the volatility of 60/40 as the target, because it provides a more appropriate comparison to traditional strategies used by institutional investors. The return of UVT strategies is particularly sensitive to the volatility target.

B.3 Trading Costs

We estimate the drag on return that stems from the turnover-induced trading required to maintain leverage targets in a strategy that levers a source portfolio S.

[6] The average difference between the 90-day T-Bill Rate and the 3-Month Euro-Dollar Deposit Rate from 1971 through 2012 was 102 basis points. So our estimate of 60 basis points was relatively conservative.

At time t, the strategy calls for an investment with a leverage ratio of λ_t. We make the harmless assumption that the value of the levered strategy at t, denoted L_t, is \$1.[7] Then the holdings in the source portfolio, or assets, are $A_t = \lambda_t$. The debt at time t is given by $D_t = \lambda_t - 1$.

We need to find holdings A_{t+1} in the portfolio at time $t+1$ that are consistent with the leverage target λ_{t+1}. This turns out to be a fixed point problem since the trading costs must come out of the investor's equity. Between times t and $t+1$, the value of the source portfolio changes from S_t to S_{t+1} and the strategy calls for rebalancing to achieve leverage λ_{t+1}. Just prior to rebalancing, the value of the investment is

$$A'_t = \lambda_t(1 + r_t^{\mathbf{S}}), \tag{B.1}$$

the liability has grown to $D'_t = (\lambda_t - 1)(1 + r_t^b)$ and the investor's equity is:

$$\begin{aligned} L'_t &= A'_t - D'_t \\ &= \lambda_t(1 + r_t^{\mathbf{S}}) - (\lambda_t - 1)(1 + r_t^b). \end{aligned} \tag{B.2}$$

Note that in formulas (B.1) and (B.2), we use the source return $r_t^{\mathbf{S}}$ *gross* of trading costs in the source portfolio.

Let $w_t = (w_{t1}, \dots, w_{tn})^\top$ denote the vector of relative weights assigned to the n asset classes in the source portfolio at time t, so that $\sum_{i=1}^n w_{ti} = 1$ for all t. Just prior to rebalancing, the weights have changed to $w'_t = (w'_{t1}, \dots, w'_{tn})^\top$, where $w'_{ti} = \frac{w_{ti}(1+r_i^t)}{1+r_t^{\mathbf{S}}}$. At time $t+1$, the strategy is rebalanced according to its rules, which produces holdings of $A_{t+1}w_{t+1}$ in the n asset classes. We let $x_t = (x_{t1}, \dots, x_{tn})^\top$ denote the vector of dollar amounts of the changes in value due to rebalancing, so that:

$$x_t = A_{t+1}w_{t+1} - A'_t w'_t. \tag{B.3}$$

If we assume a linear model, the cost of trading x_t is $\kappa\|x_t\|_1 = \sum_{i=1}^n |x_{ti}|$ for some $\kappa \geq 0$. The cost reduces the investor's equity to:

$$\begin{aligned} L_{t+1} &= L'_t - \kappa\|x_t\|_1 \\ &= \lambda_t(1 + r_t^{\mathbf{S}}) - (\lambda_t - 1)(1 + r_t^b) - \kappa\|x_t\|_1. \end{aligned} \tag{B.4}$$

Now let

$$\begin{aligned} g(\alpha) &= \frac{\alpha}{L_{t+1}} - \lambda_{t+1} \\ &= \frac{\alpha}{L'_t - h(\alpha)} - \lambda_{t+1}, \end{aligned}$$

[7]This assumption is harmless in a linear model of trading costs, which we develop here. It would be inappropriate for a realistic model of market impact.

where $g(\alpha)$ denotes the leverage implied by holding αw_{t+1} in the n assets, taking into account the effect of trading costs on equity L_{t+1}, minus the desired leverage. Assuming that g is defined on the whole interval $[0, \lambda_{t+1} L'_t]$, it is continuous, $g(0) = -\lambda_{t+1} < 0$, and $g(\lambda_{t+1} L'_t)$, so by the Intermediate Value Theorem, there exists α_{t+1} such that $g(\alpha_{t+1}) = 0$.[8] The value of α_{t+1} can readily be found by a bisection algorithm, which worked well in all of the empirical situations studied in this chapter.[9]

We set $A_{t+1} = \alpha_{t+1}$, so the holdings of the n assets are given by $A_{t+1} w_{t+1} = \alpha_{t+1} w_{t+1}$. The reduction in return due to trading costs is given by:

$$r^{\mathbf{TC}} = \kappa \| \alpha_{t+1} w_{t+1} - A'_t w'_t \|_1. \tag{B.5}$$

We compute the trading cost incurred by the source portfolio, $E[r^{\mathbf{TCS}}]$ in the same way and define the trading cost due to leverage by

$$E[r^{\mathbf{TCL}}] = E[r^{\mathbf{TC}}] - E[r^{\mathbf{TCS}}]. \tag{B.6}$$

B.4 Geometric Return

In order to analyze the effects of compounding, Booth and Fama (1992) expressed continuously compounded return in terms of arithmetic return. We have chosen to analyze the effects of compounding using the geometric average of monthly returns. Our formula (B.9) for the geometric average of monthly returns is somewhat simpler than the formula for continuously compounded return in Booth and Fama (1992). Both derivations rely on the second-order Taylor expansion approximation of the logarithm.

Let L_t denote the equity in a strategy at month t, where $t = 0, 1, \ldots, T$.

The correct ranking of realized strategy performance, taking compounding into account, is given by $G[r]$, the geometric average of the monthly returns, minus one:

$$\begin{aligned} G[r] &= \left(\frac{L_T}{L_0} \right)^{1/T} - 1 \\ &= \left[\prod_{t=0}^{T-1} \frac{L_{t+1}}{L_t} \right]^{1/T} - 1 \\ &= \left[\prod_{t=0}^{T-1} (1 + r_t) \right]^{1/T} - 1 \end{aligned} \tag{B.7}$$

[8] Typically, α_{t+1} is uniquely determined; if not, choose the largest value satisfying the equation.

[9] If there is *no* α_{t+1} such that $g(\alpha) = 0$, it means the equity of the strategy is so low that the transaction costs in getting to the desired leverage wipe out the equity. We do not observe such severe drawdown in our empirical examples, but clearly it would be possible with extreme leverage or a very volatile source portfolio.

Because the logarithm is strictly increasing, $\log(1 + G[r])$ induces exactly the same ranking of realized strategy returns as $G[r]$. It is a different ranking than the one induced by $E[r]$ and $\log(1 + E[r])$, requiring a correction term involving $\mathbf{V}ar(r)$:

$$\begin{aligned}
\log(1 + G[r]) &= \frac{1}{T}\sum_{t=0}^{T-1} \log(1 + r_t) \\
&\sim \frac{1}{T}\sum_{t=0}^{T-1} \left(r_t - \frac{(r_t)^2}{2} \right) && \text{(B.8)} \\
&= \frac{1}{T}\sum_{t=0}^{T-1} r_t - \frac{1}{T}\sum_{t=0}^{T-1} \frac{(r_t)^2}{2} \\
&= E[r] - \frac{\mathbf{V}ar(r) + (E(r))^2}{2} \\
&\sim \log(1 + E[r]) - \frac{\mathbf{V}ar(r)}{2} && \text{(B.9)} \\
G[r] &\sim (1 + E[r])\, e^{-\frac{\mathbf{V}ar(r)}{2}} - 1 && \text{(B.10)}
\end{aligned}$$

Formulas (B.8) and (B.9) approximate the logarithm by its quadratic Taylor polynomial. When $r_t > 0$, the Taylor series for logarithm is alternating and decreasing in absolute value for $|r_t| < 1$, so the error in the approximation of $\log(1 + r_t)$ in formula (B.8) is negative and bounded above in magnitude by $|r^t|^3/3$ for each month t. When $r_t < 0$, the error is positive and may be somewhat larger than $|r^t|^3/3$. Since the monthly returns are both positive and negative, the errors in months with negative returns will substantially offset the errors in months with positive returns, so the errors will tend not to accumulate over time. The approximation error in annual geometric return was at most one basis point in our risk parity examples (see table 2.4) and five basis points in our levered bond examples (see table 2.7).

B.5 Words and Formulas

Table B.9 presents the formulas accompanying the words in our Performance Attribution tables 2.2, 2.4, 2.6 and 2.7.

Table B.9: Performance Attribution

Description:	Formula:
Total Source Return (gross of trading costs)	$\mathbb{E}[\mathbf{r^S}]$ (gross of trading costs)
Leverage	$\mathbb{E}[\lambda - 1]$
Excess Borrowing Return	$\mathbb{E}[r^S - r^b]$
Levered Excess Borrowing Return	$\mathbb{E}[\lambda - 1] \cdot \mathbb{E}[r^S - r^b]$
Magnified Source Return	$\mathbb{E}[\mathbf{r^S}] + \mathbb{E}[\lambda - \mathbf{1}] \cdot \mathbb{E}[\mathbf{r^S} - \mathbf{r^b}]$
Volatility of Leverage	$\sigma(\lambda)$
Volatility of Excess Borrowing Return	$\sigma(r^S - r^b)$
Correlation(Leverage,Excess Borrowing Return)	$\rho(\lambda, r^S - r^b)$
Covariance(Leverage,Excess Borrowing Return)	$\mathbf{C}ov(\lambda, r^S - r^b)$
Source Trading Costs	$-\mathbb{E}[r^{TCS}]$
Leverage-Induced Trading Costs	$-\mathbb{E}[r^{TCL}]$
Total Levered Return (arithmetic)	$\mathbb{E}[\mathbf{r^L}]$
Compounded Arithmetic Return (gross)	$(1 + \mathbb{E}[r^L]/1200)^{12}$
Variance Correction	$\exp(-\sigma^2_{r^L}/2)$
Variance Drag	$[(1 + \mathbb{E}[r^L]/1200)^{12} \cdot \exp(-\sigma^2_{r^L}/2) - 1] \cdot 100 - \mathbb{E}[r^L]$
Approximation Error	$\mathcal{G}[\mathbf{r^L}] - [(1 + \mathbb{E}[r^L]/1200)^{12} \cdot \exp(-\sigma^2_{r^L}/2) - 1] \cdot 100$
Total Levered Return (geometric)	$\mathcal{G}[\mathbf{r^L}]$

Notes: Formulas corresponding to the words used in Performance Attribution Tables 2.2, 2.4, 2.6 and 2.7.

Appendix C

Risk Parity

C.1 Data

The results presented in this chapter are based on CRSP stock and bond data from January of 1926 through December of 2010. The aggregate stock return is the CRSP value weighted market return (including dividends) from the table *Monthly Stock - Market Indices (NYSE/AMEX/NASDAQ)* – variable name *vwretd*. The aggregate bond return is the face value outstanding (cross-sectionally) weighted average of the unadjusted return for each bond in the *CRSP Monthly Treasury (Master)* table. In this table, the variable name for the unadjusted return is *retnua* and for the face value outstanding is *iout1r*. All bonds in the table are used, provided the values for both *retnua* and *iout1r* are not missing. The value weighted market index is constructed by weighting the aggregate stock return by the total stock market value (variable name *totval*) and the aggregate bond return by the total face value outstanding of all bonds used in the return calculation. Figure 3.11 plots the stock and bond weights used to estimate the return of the value weighted index.

The proxy for the risk-free rate is the *USA Government 90-day T-Bills Secondary Market* rate, provided by Global Financial Data (http://www.globalfinancialdata.com), covering the period from January of 1926 through December of 2010. The proxy for the cost of financing leverage is the *U.S. 3-Month Euro-Dollar Deposit* rate, downloaded from the Federal Reserve (http://www.federalreserve.gov/releases/h15/data.htm). The 3-Month Euro-Dollar Deposit data is available from January of 1971 through December of 2010. Prior to January of 1971, a constant of 60 basis points is added to the 90-day T-Bill rate.[1]

[1] The average difference between the 90-day T-bill Rate and the 3-Month Euro-Dollar Deposit Rate from 1971 through 2010 is roughly 100 basis points. So our estimate of 60 basis points is relatively conservative.

C.2 Strategies

Rebalancing is monthly.

Value Weighted This is a fully invested strategy that value weights US Equity and US Treasury Bonds.

60/40 This is a fully invested strategy whose capital allocations are 60% US Equity and 40% US Treasury Bonds.

Unlevered Risk Parity This is a fully invested strategy that equalizes ex ante asset class volatilities. The volatility of each asset class is estimated at month end using a 36-month rolling window of trailing returns. The time-t estimate of volatility for asset class i is given by

$$\hat{\sigma}_{i,t} = \text{std}(r_{i,t-36}, \ldots, r_{i,t-1}).$$

The time-t portfolio weight for asset class i in the unlevered risk parity strategy is given by

$$w^u_{i,t} = \delta_t \hat{\sigma}^{-1}_{i,t},$$

where

$$\delta_t = \frac{1}{\sum_i \hat{\sigma}^{-1}_{i,t}}.$$

Levered Risk Parity This is a levered strategy that equalizes ex ante volatilities across asset classes. The leverage is chosen so that the ex post volatility matches the ex post volatility of the value weighted portfolio at each rebalancing. As in the case of the asset classes, volatility of a strategy is estimated at month end using a 36-month rolling window of trailing returns. The time t estimate of volatility for strategy s is given by

$$\hat{\sigma}_{s,t} = \text{std}(r_{s,t-36}, \ldots, r_{s,t-1}).$$

The leverage ratio required to match the trailing 36-month realized volatility of the value weighted index is the quotient of the volatility estimate for the value weighted portfolio, $\hat{\sigma}_{v,t}$, by the volatility estimate for the and unlevered risk parity portfolio, $\hat{\sigma}_{u,t}$:

$$l_t = \frac{\hat{\sigma}_{v,t}}{\hat{\sigma}_{u,t}}.$$

The time-t portfolio weight for asset class i at time t in the levered risk parity strategy is given by

$$w_{l,i,t} = l_t w_{u,i,t}.$$

The return of the levered risk parity portfolio at time t is

$$\begin{aligned} r_{l,t} &= \sum_i w_{u,i,t} r_{i,t} + \sum_i (l_t - 1) w_{u,i,t}(r_{i,t} - r_{b,t}) \\ &= \sum_i w_{u,i,t} r_{i,t} + \sum_i (w_{l,i,t} - w_{u,i,t})(r_{i,t} - r_{b,t}), \end{aligned}$$

where $r_{b,t}$ is the borrowing rate at time t.

Asness, Frazzini and Pedersen (2012) implement an unconditional levered risk parity strategy. The asset class weights in this strategy depend on a time-independent scale factor k chosen so that volatility of excess returns estimated over the entire sample, 1926–2010, matches the volatility of excess returns of the value weighted strategy. To be precise,

$$w_{l.unc,i,t} = k\hat{\sigma}_{i,t}^{-1},$$

$$r^e_{l.unc,t} = r_{l.unc} - r_{f,t},$$

and

$$\sigma = \text{std}(r^e_{l.unc,37}, \ldots, r^e_{l.unc,T}), \tag{C.1}$$

where σ is a desired target volatility, (which Asness, Frazzini and Pedersen (2012) set to be the realized volatility of the value-weighted portfolio). Here, T is the *last* month in the sample period (i.e. if the sample period is January 1926 through December 2010, then $T = 1020$). Note that the target σ is not known until the end of the period. Moreover, even if σ were set to some constant that were known in 1926, k cannot be computed until the full history through 2010 is known. If k and σ were set to some constants in 1926, then equation C.1 would not be satisfied. Thus, this version of the unconditional levered risk parity is not investable.

The conditional and unconditional levered risk parity strategies differ in other important ways. Consider, for example, their responses to an upward spike in equity volatility. All else equal, both strategies will increase the ratio of capital in bonds to capital in equity, but the conditional strategy will increase its leverage, while the unconditional strategy will decrease its leverage.

C.3 Trading Costs

To estimate trading costs due to turnover, we need to express the change in portfolio weights due to price movements (or returns) over a single period. For any strategy, the time-t return-modified weight to asset i is given by

$$\tilde{w}_{i,t} = \frac{(1 + r_{i,t})w_{i,t-1}}{\sum_j (1 + r_{j,t})w_{j,t-1}}$$

and the turnover required to rebalance the strategy is given by

$$x_t = \sum_j |\tilde{w}_{j,t} - w_{j,t}|.$$

In in view of the large and variable leverage implicit in our levered risk parity strategy, we explicitly show the impact of leverage on turnover.

$$x_t = \sum_j |\tilde{w}_{u,j,t}\ell_{t-1} - w_{u,j,t}\ell_t|.$$

Trading costs at time t are then given by

$$c_t = x_t z_t,$$

where (by assumption) z_t is equal to 1% for 1926-1955, 0.5% for 1956-1970, and 0.1% for 1971-2010, and trading cost adjusted returns are given by

$$r'_{l,t} = r_{l,t} - c_t.$$

C.4 Bootstrap Estimates

In order to reflect the empirical properties of our data, we use a bootstrap to estimate the P-values in table 3.1. For a given strategy and evaluation period, suppose we have a sample of T monthly observations of excess return. The excess return reported in table 3.1 is the annualized mean. To estimate the P-value for the excess return, we draw 10,000 bootstrap samples of T observations (with replacement) from the empirical distribution. We calculate the mean of each bootstrap sample. The P-value is given by:

$$P = \frac{\text{\#means} <= 0}{N}.$$

The bootstrap procedure for the alpha P-value is different. Suppose

$$R_s = \alpha + \beta R_b + \epsilon,$$

where R_b is the vector of excess returns of a benchmark portfolio (i.e. $R_b = (R_{b,1}, \ldots, R_{b,T})'$), which in our case is the value weighted portfolio, and R_s is the vector of excess returns of a strategy portfolio. A time series regression to estimate alpha and beta generates the residuals:

$$e_t = R_{s,t} - \hat{\alpha} - \hat{\beta} R_{b,t},$$

for $t = 1, \ldots, T$. Next, we draw 10,000 samples (with replacement) of T observations from the empirical distribution of residuals and, for each sample, regenerate the strategy returns as:

$$R_s^* = \hat{\alpha} + \hat{\beta} R_b + \epsilon^*,$$

where ϵ^* is the vector of resampled residuals. Then for each sample, we run a time series regression based on the equation above to get new estimates of alpha ($\hat{\alpha}^*$) and beta ($\hat{\beta}^*$). The P-value for alpha is given by:

$$P = \frac{\#\hat{\alpha}^* <= 0}{N}.$$

The probability estimates in section 3.4are also based on a bootstrap. For example, to calculate the probability that 60/40 will outperform levered risk parity over a 20 year horizon, we draw $10,000$ samples of 240 contemporaneous monthly observations from empirical distribution of the total returns to the 60/40 and levered risk parity portfolios. For each sample, we calculate the cumulative return to each strategy over the 20 year horizon and record the difference $cr_d = cr_{rp} - cr_{60/40}$. The probability estimate is given by:

$$\mathbb{P} = \frac{\#cr_d < 0}{N}.$$

C.5 Two Robustness Checks

We consider the impact of two of the assumptions that are made in this study.

C.5.1 Rebalancing Horizon

The monthly rebalancing horizon used in our studies is shorter than the horizon typically used by pensions, endowments, and other long-term investors. Figure 3.12 shows the cumulative return to the four strategies when they are rebalanced annually. The left panel show the base case where borrowing is at the 90-Day T-Bill Rate and no adjustment is made for trading due to turnover. In the right panel, borrowing is at the 3-Month Euro-Dollar Deposit Rate and adjustments are made for trading due to turnover. The results are less dramatic than they are when rebalancing monthly horizon, but they are qualitatively similar.

C.5.2 Borrowing Cost Assumptions

The US 3-Month Euro-Dollar Deposit Rates used, starting in 1971, as the estimate of the implicit interest rate when levering through futures. In the prior period 1926–1970 when the 3-Month Euro-Dollar Deposit Rate was not available, we extrapolate borrowing costs to be the 90-Day T-Bill Rate plus 60 basis points. This is a conservative estimate since the average spread between the3-Month Euro-Dollar Deposit Rate and the 90-Day T-Bill Rate during the period 1971–2010 is roughly 100 basis points. Table C.2 shows the impact of varying the extrapolated spread on strategy performance. We consider spreads ranging between 25 and 125 basis points. After adjustment for turnover-induced trading, the risk premium of levered risk parity over the market had low statistical significance even when a level of 25 basis points is taken to be the spread between the 3-Month Euro-Dollar Deposit Rate and the 90-Day T-Bill Rate between 1926 and 1970.

Table C.2: Effect of Alternate Borrowing Cost Assumptions for the Pre-1971 Period

Pre-1971 Borrowing Cost Spread Over T-bills (bp)	Excess Return	*P*-value Excess Return	Alpha	*P*-value Alpha	Volatility	Sharpe Ratio	Skewness	Excess Kurtosis
A. Long sample, adjusted for 3M-EDR (RP - value-weighted)								
25	1.54	0.10	2.22	0.03	10.72	0.14	-0.54	12.41
50	1.17	0.16	1.85	0.05	10.72	0.11	-0.56	12.48
60	1.03	0.20	1.70	0.07	10.72	0.10	-0.57	12.50
75	0.81	0.25	1.48	0.10	10.73	0.08	-0.58	12.54
100	0.44	0.36	1.11	0.17	10.73	0.04	-0.61	12.59
125	0.07	0.47	0.74	0.26	10.74	0.01	-0.63	12.64
B. Long sample, adjusted for 3M-EDR (RP - 60/40)								
25	0.80	0.23	0.50	0.33	10.10	0.08	-1.11	13.53
50	0.43	0.35	0.13	0.45	10.11	0.04	-1.14	13.64
60	0.29	0.39	-0.02	0.50	10.11	0.03	-1.15	13.68
75	0.06	0.47	-0.24	0.58	10.11	0.01	-1.16	13.74
100	-0.30	0.60	-0.61	0.70	10.12	-0.03	-1.19	13.84
125	-0.67	0.73	-0.97	0.81	10.13	-0.07	-1.22	13.93
C. Long sample, adjusted for 3M-EDR and Trading Costs (RP - value-weighted)								
25	0.63	0.29	1.31	0.13	10.74	0.06	-0.64	12.99
50	0.26	0.41	0.94	0.21	10.74	0.02	-0.66	13.04
60	0.11	0.46	0.79	0.25	10.75	0.01	-0.67	13.06
75	-0.11	0.53	0.57	0.31	10.75	-0.01	-0.68	13.09
100	-0.48	0.66	0.20	0.43	10.76	-0.04	-0.70	13.13
125	-0.84	0.76	-0.16	0.56	10.77	-0.08	-0.73	13.17
D. Long sample, adjusted for 3M-EDR and Trading Costs (RP - 60/40)								
25	0.01	0.50	-0.29	0.60	10.12	0.00	-1.18	13.84
50	-0.36	0.64	-0.66	0.73	10.13	-0.04	-1.21	13.94
60	-0.51	0.67	-0.81	0.77	10.13	-0.05	-1.22	13.98
75	-0.73	0.74	-1.03	0.82	10.14	-0.07	-1.24	14.03
100	-1.10	0.84	-1.40	0.89	10.15	-0.11	-1.26	14.11
125	-1.47	0.91	-1.77	0.94	10.16	-0.14	-1.29	14.18

Notes: Impact of the borrowing cost extrapolation on the risk premium of levered risk parity over the value weighted strategy. 1926–2010. In the top 2 panels, the levered risk parity strategy is financed at the 3-Month Euro-Dollar Deposit Rate. In the bottom 2 panels, an additional adjustment is made for trading due to turnover.

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