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Modular Forms Associated to Real Cubic Fields

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ABSTRACT OF THE DISSERTATION

Modular Forms Associated to Real Cubic Fields

by

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Doctor of Philosophy in Mathematics
University of California, Los Angeles, 2018
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In the 1970s Don Zagier introduced a family of Hilbert modular forms for real quadratic fields and a related family of elliptic modular forms. These forms possess a number of remarkable properties, and have inspired a great deal of research over the ensuing decades. After reviewing the necessary background material and introducing Zagier's work, this dissertation presents candidates for a generalization of these forms to real cubic fields, and studies some of their properties.

The dissertation of Joseph Ford Hughes is approved.

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CHAPTER 1

Introduction

In the early 1970s Hirzebruch and Zagier, following a suggestion of Serre, began to investigate intersection numbers on modular surfaces. Their research led to the seminal paper [HZ76], and a few years later led Zagier to define the Hilbert modular forms

$$\omega_m(z_1, z_2) = \sum_{\substack{a,b\in\mathbb{Z},\lambda\in\mathfrak{d}^{-1} \\ \lambda\lambda'-ab=m/D}}{}^{'} (az_1z_2 + \lambda z_1 + \lambda' z_2 + b)^{-k}$$

defined for quadratic fields $K = \mathbb{Q}(\sqrt{D})$ with $D > 0$. These remarkable forms lead to an explicit kernel function for the celebrated Doi-Naganuma lift for quadratic fields. Specifically, if we define

$$\Omega(z_1, z_2; \tau) = \sum_{m=1}^{\infty} m^{k-1}\omega_m(z_1, z_2)e^{2\pi i m\tau}$$

then (for z_1 and z_2 fixed) Ω is a cusp form for $\Gamma_0(D)$ with quadratic character ε, and for each cusp form f for $\Gamma_0(D)$ and ε, the map sending f to its Petersson inner product $(f, \Omega)_\tau$ is up to a constant factor the Doi-Naganuma lift of f.

Zagier also considered the elliptic modular forms defined by restricting the ω_m to the diagonal $z_1 = z_2$. The result is an infinite sum of so-called hyperbolic Eisenstein series

$$f_k(\Delta, z) = \sum_{\substack{a,b,c\in\mathbb{Z} \\ b^2-4ac=\Delta}}{}^{'} (az^2 + bz + c)^{-k}$$

These forms, originally considered simply "because they were there", turn out to be as remarkable as the ω_m themselves. Indeed, they can be used to define a kernel function for the Shimura lift from half-integer weight forms, and have rational period functions. The ω_m and the hyperbolic Eisenstein series f_k have generated a great deal of interest over the ensuing decades, most recently in an extension of the f_k to negative discriminants in [Ben15].

This dissertation presents candidates for a generalization of the ω_m and f_k to real cubic fields. The next two chapters give some general background, and the fourth chapter explains the properties of the ω_m in detail, following [Zag75]. Chapters five and six contain nearly all of the new results. Chapter five is devoted to a generalization of the ω_m to real cubic fields, and chapter six to a generalization of the f_k.

In chapter five the main innovation is the use of the hyperdeterminant, a generalization of the determinant which was first considered by Cayley [Cay09] and then fell into obscurity for nearly 150 years until it was resurrected by Gelfand et. al. in [GKZ08]. Hyperdeterminants remain largely unknown even after Gelfand's work, perhaps because the number of terms grows so quickly with the dimension of the hypermatrix, and perhaps because hypermatrices themselves are still relatively uncommon despite their growing importance in areas as diverse as invariant theory and statistics. The hyperdeterminant of a hypermatrix is nevertheless a natural generalization of the determinant of a matrix, and enjoys many analogous properties. The $2 \times 2 \times 2$ hypermatrix in particular has an interesting geometric interpretation, as we will see.

After defining the generalization $H_m(z_1, z_2, z_3)$ of Zagier's Hilbert modular forms, we will prove that they are well-defined (which is not immediately obvious from the definition), that they converge absolutely, and finally that they are Hilbert modular forms. We also show that when $m = 0$, $H_0(z_1, z_2, z_3)$ is a multiple of the Hecke-Eisenstein series for the cubic field K. Unfortunately, many of the computations which are merely very difficult in the quadratic case become (to date) insurmountably difficult in the cubic case, and so the question of whether the H_m are in fact a suitable generalization of the ω_m is still unresolved, although the similarities are striking.

In chapter six, the natural generalization

$$h_{k,\Delta}(z) = \sum_{\substack{a,b,c,d \in \mathbb{Z} \\ \mathrm{disc}(a,b,c,d)=\Delta}} (az^3 + bz^2 + cz + d)^{-k}$$

of the f_k is much easier to define, and is also a bit more amenable to computations. Two related expressions are given for the Fourier coefficients. Just as for the hyperbolic Eisenstein series f_k, these Fourier series consist of the sum of an exponential sum together with an elementary function. In the case $\Delta = 0$ we prove that $h_{k,0}$ (with the definition suitably modified) is a scalar multiple of the weight $3k$ Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$. As the vast majority of the work on modular forms of non-integral weight has been devoted to the half-integer case, it is not yet clear whether the h_k have any arithmetic significance. But once again the similarities to the f_k are striking and encouraging.

CHAPTER 2

Background: Elliptic Modular Forms

This chapter summarizes the fundamental definitions and results for modular forms in one variable. The proofs of these results can be found in Miyake [Miy06] or Serre [Ser73].

2.1 Discrete Subgroups of $\mathrm{SL}_2(\mathbb{R})$

Let $\mathbb{H} = \{z = x + iy : y > 0\}$ be the upper half complex plane. The group $\mathrm{SL}_2(\mathbb{R})$ of 2×2 matrices with real entries having determinant 1 acts on $\mathbb{H}$ via Möbius transformations as follows: if

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and $z \in \mathbb{H}$, then

$$\gamma z = \frac{az + b}{cz + d} \tag{2.1}$$

It is not hard to check that

$$\Im(\gamma z) = \frac{\Im z}{|cz + d|^2} \tag{2.2}$$

and that $\gamma_1(\gamma_2 z) = (\gamma_1\gamma_2)z$, and so (2.1) does in fact define an action. Moreover, it follows that γ also maps $\mathbb{R} \cup \{\infty\}$ into itself, where we define $\gamma(\infty) = \infty$ if $c = 0$, and otherwise $\gamma(\infty) = \frac{a}{c}$. Note that γ and $-\gamma$ define the same Möbius transformation.

The group $\mathrm{SL}_2(\mathbb{R})$ can be made into a topological group by giving it the subspace topology from $\mathrm{M}_2(\mathbb{R}) \simeq \mathbb{R}^4$, and a subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$ is said to be discrete if it is discrete with respect to this topology. One can show that Γ is discrete if and only if its action on $\mathbb{H}$ is discontinuous, meaning that

$$\#\{\gamma \in \Gamma : \gamma(K) \cap K \neq \emptyset\} < \infty$$

for every compact subset K of $\mathbb{H}$.

The elements of $\mathrm{SL}_2(\mathbb{R})$ can be grouped into three classes: if $|\mathrm{tr}(\gamma)| < 2$ then γ is said to be elliptic, if $|\mathrm{tr}(\gamma)| = 2$ then γ is parabolic, and if $|\mathrm{tr}(\gamma)| > 2$ then γ is said to be hyperbolic.

The distinction between these classes is perhaps better understood in terms of fixed points: let

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be an element of $\mathrm{SL}_2(\mathbb{R})$. If $c \neq 0$, then the equation

$$z = \gamma z = \frac{az+b}{cz+d}$$

has solutions

$$z = \frac{a-d \pm \sqrt{(a+d)^2-4}}{2c} \tag{2.3}$$

From (2.3) we see that if γ is elliptic then γ has exactly one fixed point in $\mathbb{H}$, if γ is hyperbolic then γ has two fixed points in $\mathbb{R}$, and if γ is parabolic with $c \neq 0$ then γ has exactly one fixed point in $\mathbb{R}$.

If $c = 0$ then γ is parabolic if and only if $a = d = \pm 1$, in which case γ has ∞ as its only fixed point. Otherwise γ is hyperbolic and fixes ∞ and $\frac{b}{d-a}$.

Elliptic and parabolic fixed points are especially important because they must be handled carefully when defining a Riemann surface structure on the quotient $\Gamma\backslash\mathbb{H}$.

Suppose that $\zeta \in \mathbb{H}$ is a fixed point for an elliptic element of a discrete subgroup Γ. What is the structure of the stabilizer $\Gamma_\zeta = \{\gamma \in \Gamma : \gamma\zeta = \zeta\}$? The map $\zeta \mapsto \frac{z-\zeta}{z+\zeta}$ is a biholomorphic map from $\mathbb{H}$ to the unit disk $\mathbb{D}$, and using this map we can identify Γ with a discrete subgroup of the group of automorphisms of $\mathbb{D}$, with the elements of Γ_ζ mapping to automorphisms which fix 0.

It follows from the Schwarz lemma that the only automorphisms of $\mathbb{D}$ fixing 0 are the rotations. Since any discrete subgroup of $\mathrm{SO}(2)$ is finite cyclic, Γ_ζ must be a finite cyclic group.

Similarly, suppose that $x \in \mathbb{R} \cup \{\infty\}$ is a parabolic fixed point for Γ, and let Γ_x be the stabilizer of x in Γ. First suppose that $x = \infty$. Since Γ is discrete it follows that all of the elements of Γ_∞ must be parabolic, and that there is some $h > 0$ such that

$$\{\pm I\}\Gamma_\infty = \left\{ \pm \begin{bmatrix} 1 & nh \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\} \tag{2.4}$$

If $x \in \mathbb{R}$ then the matrix $\sigma_x = \begin{bmatrix} 0 & 1 \\ -1 & x \end{bmatrix}$ maps x to ∞, and it follows that $\sigma_x^{-1}\Gamma_x\sigma_x$ has the form above.

A cusp of Γ is a Γ-equivalence class of parabolic fixed points for Γ, and the width of the cusp is defined to be the real number h such that (2.4) holds.

The upper half-plane $\mathbb{H}$ can be made into a Riemannian manifold by giving it the $\mathrm{SL}_2(\mathbb{R})$-invariant metric $ds^2 = y^{-2}(dx^2 + dy^2)$. This is the familiar Poincaré half-plane model for hyperbolic geometry. One can also check that the measure $d\mu(z) = y^{-2}dxdy$ $(z = x + iy)$ is $\mathrm{SL}_2(\mathbb{R})$-invariant.

Given a discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})$, let $\mathbb{H}^*$ consist of $\mathbb{H}$ and the set of cusps of Γ. We define a topology on $\mathbb{H}^*$ as follows: the induced topology on $\mathbb{H}$ is the usual one, and $\mathbb{H}$ is a open dense subset of $\mathbb{H}^*$. If x is a cusp of Γ and σ_x is defined as above, then a basis for the open neighborhoods of x is given by

$$\sigma_x^{-1}(U_C) \cup \{x\}$$

where

$$U_C = \{z \in \mathbb{H} : \Im(z) > C\} \tag{2.5}$$

With this definition, the quotient $\Gamma\backslash\mathbb{H}^*$ is a locally compact Hausdorff space, and the inclusion $\Gamma\backslash\mathbb{H} \to \Gamma\backslash\mathbb{H}^*$ is an open embedding. For the subgroups Γ we are interested in, $\Gamma\backslash\mathbb{H}^*$ will in fact be compact. One can make $\Gamma\backslash\mathbb{H}^*$ into a Riemann surface with a little more work, though some care must be taken with the elliptic fixed points of Γ.

2.2 The modular group

The modular group is the subgroup $\mathrm{SL}_2(\mathbb{Z})$ of $\mathrm{SL}_2(\mathbb{R})$ consisting of matrices with integer entries. It is generated by the matrices

$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{2.6}$$

Every $z \in \mathbb{H}$ can be transformed by Γ to a point in the domain

$$\mathcal{F} = \{z \in \mathbb{H} : -\frac{1}{2} \leq \Re z \leq \frac{1}{2}, |z| \geq 1\} \tag{2.7}$$

and this point is unique with the exception of the two sides $\Re z = \pm\frac{1}{2}$ (which are equivalent under T) and the two "halves" of the unit semicircle (which are equivalent under S). The modular group has a single cusp at ∞, and the only elliptic fixed points (up to $\mathrm{SL}_2(\mathbb{Z})$-equivalence) are i and $\rho = e^{\frac{2\pi i}{3}}$. The quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^*$ can be made into a compact Riemann surface.

Besides the modular group itself, certain subgroups of $\mathrm{SL}_2(\mathbb{Z})$ are of interest. For each positive integer N, define the three "principal congruence subgroups"

$$\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, d \equiv 1, b, c \equiv 0 \mod N \right\} \tag{2.8}$$

$$\Gamma_1(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, d \equiv 1, c \equiv 0 \mod N \right\} \tag{2.9}$$

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : c \equiv 0 \mod N \right\} \tag{2.10}$$

Note that $\Gamma(N) \subset \Gamma_1(N) \subset \Gamma_0(N)$ for each N, and when $N = 1$ these three subgroups are the full modular group. While it is possible to consider more general congruence subgroups, these three will be sufficient for our purposes (and in fact we will mainly be interested in $\Gamma_0(N)$).

2.3 Modular forms

Let k be a positive integer, and let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic function. We say that f is a modular form of weight k for $\mathrm{SL}_2(\mathbb{Z})$ if

$$f(\gamma z) = (cz+d)^k f(z)$$

for all

$$\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

and if it is holomorphic at infinity in the following sense:

Taking $\gamma = T$, we see that any function satisfying $f(\gamma z) = (cz+d)^k f(z)$ is periodic of period 1, and hence has a Fourier expansion

$$f(z) = \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n z}$$

f is said to be holomorphic at infinity if $c_n = 0$ for all $n < 0$. If in addition $c_0 = 0$, then f is a cusp form. The condition $f(\gamma z) = (cz+d)^k f(z)$ is sometimes rewritten in the following ways: either as

$$f(\gamma z) = j(\gamma, z)^k f(z)$$

where $j(\gamma, z) = cz + d$ is the "factor of automorphy", or as $f|_k\gamma = f$, where

$$(f|_k\gamma)(z) = (cz+d)^{-k} f(\gamma z) \tag{2.11}$$

Modular forms for congruence subgroups can be defined similarly, though the definition becomes a bit more complicated because these groups often have more than one cusp.

We will also need a slight generalization of the definition above: if Γ is a congruence subgroup of the modular group and $\chi : \Gamma \to \mathbb{C}^\times$ is a character of finite order, a modular form of weight k for Γ and χ satisfies

$$f(\gamma z) = \chi(\gamma)(cz+d)^k f(z) \tag{2.12}$$

for all $\gamma \in \Gamma$, and also satisfies the appropriate growth conditions at the cusps of Γ. Most commonly $\Gamma = \Gamma_0(N)$, and $\chi(\gamma) = \chi_N(d)$ for χ_N some Dirichlet character mod N (d is invertible mod N if $\gamma \in \Gamma_0(N)$). If $-I \in \Gamma$, then there will be no non-zero modular forms of weight k for Γ and χ unless $\chi(-I) = (-1)^k$.

What is the motivation for the definition of modular forms? As mentioned above, the quotient $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ (together with a cusp at infinity) can be made into a compact Riemann surface, and then the two conditions in the definition of a modular form of weight k are precisely what are needed for f to define a holomorphic differential of weight k on this quotient. The complex vector space of such differentials is finite dimensional, and the dimension can be explicitly computed using Riemann-Roch.

On the other hand, there are many examples of modular forms whose Fourier coefficients involve interesting arithmetic functions (we will see a few examples in the next section) and since the vector space of modular forms of weight k is finite-dimensional this leads to identities relating these functions.

There is another reason to study modular forms: if $f(z)$ is a modular form of weight k with Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} c_n e^{2\pi i n z}$$

then define the L-function of f to be the Dirichlet series

$$L(s, f) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}$$

These L-functions turn out to have analytic continuations and functional equations, as will be explained in more detail in the next chapter. Moreover, there are converse theorems which state that Dirichlet series which have “enough” functional equations must be the L-function of a modular form.

These results have two important consequences: first, L-functions can be used to understand modular forms, and second, if some other object with an L-function (e.g. an elliptic curve) can be shown to correspond to a modular form in some sense, then its L-function will “inherit” the desirable properties of the L-function of the corresponding modular form.

2.4 Eisenstein series and Poincaré series

It is not immediately obvious from the definition that there should be any interesting examples of modular forms. In this section we will introduce a construction that leads to important families of modular forms. This construction encompasses the familiar case of Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$, but we will work in greater generality to fit the needs of a later chapter.

Fix an integer $k \geq 3$, and let Γ be the modular group or one of its congruence subgroups, and χ a character of Γ of finite order. We also suppose that $\chi(-I) = (-1)^k$ if Γ contains $-I$.

Let Λ be a subgroup of Γ, and suppose that $\phi : \mathbb{H} \to \mathbb{C}$ is a holomorphic function such that

$$\phi|_k\lambda = \chi(\lambda)\phi$$

for all $\lambda \in \Lambda$. If $\kappa_1, \ldots, \kappa_r$ are representatives of the Γ-inequivalent cusps of Γ, then let $V_1, \ldots, V_r$ be open neighborhoods of $\kappa_1, \ldots, \kappa_r$, and define

$$\mathbb{H}' = \mathbb{H} \setminus \Big(\bigcup_{j=1}^{r} \bigcup_{\gamma \in \Gamma} \gamma V_j \Big) \tag{2.13}$$

Finally, suppose that

$$\int_{\Lambda\setminus\mathbb{H}'} |\phi(z)| \Im(z)^{\frac{k}{2}} \, d\mu(z) < \infty \tag{2.14}$$

Then the series

$$F(z) = F_k(z; \phi, \chi, \Lambda, \Gamma) = \sum_{\gamma \in \Lambda\setminus\Gamma} \overline{\chi(\gamma)} (\phi|_k\gamma)(z) \tag{2.15}$$

is absolutely and locally uniformly convergent on $\mathbb{H}$, and moreover $F|_k\gamma = \chi(\gamma)F$ for all $\gamma \in \Gamma$.

We need two additional conditions to guarantee that F will be holomorphic at infinity: let κ be a cusp of Γ, and σ an element of $\mathrm{SL}_2(\mathbb{R})$ such that $\sigma\kappa = \infty$. If κ is not a cusp of Λ, we suppose that there are $\varepsilon > 0$ and $M, c > 0$ such that

$$|(\phi|_k\sigma^{-1})(z)| \leq M|z|^{-1-\varepsilon}$$

for all z with $\Im z \geq c$, and if κ is a cusp of Λ then we suppose that there are $\varepsilon \geq 0$ and $M, c > 0$ such that

$$|(\phi|_k\sigma^{-1})(z)| \leq M|z|^{-\varepsilon}$$

for all z with $\Im z \geq c$.

With these additional requirements, F will be holomorphic at the cusps of Γ, and if the ε in the second condition can be taken to be positive then F will vanish at the corresponding cusp.

This construction is quite general, but one important special case is the following: let κ be a cusp of Γ, and Γ_κ the stabilizer of κ in Γ. Choose $\sigma \in \mathrm{SL}_2(\mathbb{R})$ such that $\sigma\kappa = \infty$, and let h be a positive real number such that

$$\sigma\Gamma_\kappa\sigma^{-1}\{\pm I\} = \left\{ \pm \begin{bmatrix} 1 & nh \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}$$

and define

$$\phi_m(z) = j(\sigma, z)^{-k} \exp\left(\frac{2\pi i m \sigma z}{h}\right) \tag{2.16}$$

for an integer $m \geq 0$. Suppose also that $\chi(\gamma)j(\sigma\gamma\sigma^{-1}, z)^k = 1$ for all $\gamma \in \Gamma_x$. Then the Poincaré series F defined using ϕ_m is a modular form of weight k for Γ and χ, and is a cusp form if $m \geq 1$. If $m = 0$ then F is called an Eisenstein series.

If $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ with the trivial character and $\kappa = \infty$ then we can take $\sigma = I$, and since $h = 1$ we obtain

$$F_k(z; \phi_m, 1, \Gamma_\infty, \Gamma) = \sum_{\Gamma_\infty \backslash \Gamma} j(\gamma, z)^{-k} e^{2\pi i m z}$$

In particular, if $m = 0$ then

$$F_k(z; \phi_0, 1, \Gamma_\infty, \Gamma) = \sum_{\Gamma_\infty \backslash \Gamma} j(\gamma, z)^{-k}$$

which is (up to a normalizing factor) the usual Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$.

2.4.1 Poincaré series and Fourier expansions

Suppose that $k \geq 3$, and that f is a cusp form of weight k for Γ and χ. Fix a cusp κ for Γ. If once again σ is an element of $\mathrm{SL}_2(\mathbb{R})$ such that $\sigma\kappa = \infty$, then f has a Fourier expansion

$$(f|_k\sigma^{-1})(z) = \sum_{n=1}^{\infty} a_n e^{\frac{2\pi i n z}{h}} \tag{2.17}$$

where h is the width of the cusp κ as defined in section 2.1.

This Fourier expansion is closely related to Poincaré series, as follows: let ϕ_m be as in (2.16), and define

$$g^{(m)}(z) = F_k(z; \phi_m, \chi, \Gamma_\kappa, \Gamma) \tag{2.18}$$

Then the Petersson inner product of f and $g^{(m)}$ is

$$\int_{\Gamma\backslash\mathbb{H}} f(z)\overline{g^{(m)}(z)}\Im(z)^k \, d\mu(z) = a_m(4\pi m)^{1-k}h^k(k-2)! \tag{2.19}$$

for $m \geq 1$, and is zero when $m = 0$. This can be verified using the usual "unfolding" trick for the Poincaré series.

The Fourier expansions of the $g^{(m)}$ at a cusp κ will be needed later, and for this we will make some simplifying assumptions: set $\Gamma = \Gamma_0(D)$ and suppose that we want to determine the Fourier expansion of $g_\kappa^{(m)}$ at ∞. Also assume that the width of the cusp ∞ is 1. Hence

$$g_\kappa^{(m)}(z) = \sum_{n=1}^{\infty} g_{m,n}^{\kappa} e^{2\pi i n z} \tag{2.20}$$

and the problem is to determine the coefficients $g_{m,n}^{\kappa}$.

Begin with the following lemma:

Lemma 2.4.1. *If* $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R})$ *and* $t > 0$*, then*

$$\sum_{r=-\infty}^{\infty} (c(z+r)+d)^{-k} e^{2\pi i t \frac{a(z+r)+b}{c(z+r)+d}} = \frac{2\pi i^k}{c} \sum_{m=1}^{\infty} \left(\frac{m}{t}\right)^{\frac{k-1}{2}} J_{k-1}\left(\frac{4\pi}{c}\sqrt{mt}\right) e^{\frac{2\pi i}{c}(ta+md)} e^{2\pi i m z} \tag{2.21}$$

where $J_{k-1}(z)$ *is a Bessel function.*

For the proof, first observe that if the result holds for $\gamma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, i.e. if

$$\sum_{r=-\infty}^{\infty} (z+r)^{-k} e^{-\frac{2\pi it}{z+r}} = 2\pi i^k \sum_{m=1}^{\infty} \left(\frac{m}{t}\right)^{\frac{k-1}{2}} J_{k-1}(4\pi\sqrt{mt}) e^{2\pi imz} \tag{2.22}$$

then the general case will follow by replacing z with $z + \frac{d}{c}$, t with $\frac{t}{c^2}$, and multiplying both sides by $c^{-k} e^{\frac{2\pi ita}{c}}$.

So it is enough to prove the second identity. To this end, note that the series on the left converges absolutely and locally uniformly in $\mathbb{H}$, and is periodic with period 1, hence has a Fourier expansion

$$\sum_{m=1}^{\infty} c_m e^{2\pi imz}$$

with

$$c_m = \int_{iC}^{iC+1} \Big(\sum_{r=-\infty}^{\infty} (z+r)^{-k} e^{-\frac{2\pi it}{z+r}} \Big) e^{-2\pi imz} \, dz$$

Interchanging the sum and integral and making a change of variables, we get

$$c_m = \int_{iC-\infty}^{iC+\infty} z^{-k} e^{-2\pi i\frac{t}{z}} e^{-2\pi imz} \, dz = 2\pi i^k \left(\frac{m}{t}\right)^{\frac{k-1}{2}} J_{k-1}(4\pi\sqrt{mt})$$

which proves the lemma.

Now let us return to the Poincare series $g_\kappa^{(m)}$, which we will now write as

$$g_\kappa^{(m)}(z) = \frac{1}{2} \sum_{\Gamma_\kappa \backslash \sigma\Gamma} \chi(\sigma^{-1}\gamma) j(\gamma, z)^{-k} e^{\frac{2\pi im\gamma z}{h}} \tag{2.23}$$

where $\sigma\kappa = \infty$,

$$\Gamma_\kappa = \sigma\Gamma\sigma^{-1} \cap \Big\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \Big\} \tag{2.24}$$

and h is the width of the cusp κ.

Now two matrices $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\gamma' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$ in $\sigma\Gamma$ are Γ_κ-equivalent if and only if $(c', d') = (c, d)$, and a pair (c, d) appears as the second row of some γ if and only if c, d are coprime and $-\frac{d}{c}$ is Γ-equivalent to κ. Hence

$$g_\kappa^{(m)}(z) = \frac{1}{2} \sum_{\substack{\gcd(c,d)=1 \\ \frac{-d}{c} \sim \kappa}} \chi\Big(\sigma^{-1} \begin{bmatrix} a & b \\ c & d \end{bmatrix}\Big) (cz+d)^{-k} e^{\frac{2\pi imz}{h} \frac{az+b}{cz+d}} \tag{2.25}$$

This sum contains a term with $c = 0$ if and only if $\kappa = \infty$, in which case $d = \pm 1$, $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \pm I$, and $h = 1$. So the contribution from $c = 0$ is $\delta_{\kappa\infty} e^{2\pi imz}$.

Otherwise we can assume $c > 0$ (and multiply by 2), and then since Γ contains the subgroup

$$\Big\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \Big\}$$

it follows that if a matrix with second row (c, d) lies in $\sigma\Gamma$ then so does a matrix with second row $(c, d + rc)$ for any $r \in \mathbb{Z}$. Hence the condition $-\frac{d}{c} \sim \kappa$ depends only on d mod c, and so the contribution to $g_\kappa^{(m)}$ of terms with $c > 0$ is

$$\sum_{c=1}^{\infty} \sum_{\substack{d \mod c \\ \gcd(c,d)=1 \quad -\frac{d}{c} \sim \kappa}} \chi\Big(A_P^{-1} \begin{bmatrix} a & b \\ c & d \end{bmatrix}\Big) \sum_{r \in \mathbb{Z}} [c(z+r)+d]^{-k} e^{\frac{2\pi im}{h} \frac{a(z+r)+b}{c(z+r)+d}} \tag{2.26}$$

Using the lemma, we therefore obtain

$$g_{m,n}^\kappa = \delta_{\kappa\infty}\delta_{mn} + 2\pi i^k \Big(\frac{nh}{m}\Big)^{\frac{k-1}{2}} \sum_{c=1}^{\infty} J_{k-1}\Big(\frac{4\pi}{c}\sqrt{\frac{mn}{h}}\Big) \sum_{\substack{d \mod c \\ gcd(c,d)=1 \\ -\frac{d}{c} \sim \kappa}} \chi\Big(\sigma^{-1} \begin{bmatrix} a & b \\ c & d \end{bmatrix}\Big) e^{\frac{2\pi i}{c}(n\frac{a}{h} + md)} \tag{2.27}$$

CHAPTER 3

Background: Hilbert modular forms

In this chapter we provide some background on Hilbert modular forms which will be needed for the next two chapters. The results which are stated without proof can be found in either Bump [Bum97] or Freitag [Fre90]. We also summarize some important results on the L-function of a modular form, following Miyake [Miy06].

3.1 Discrete subgroups of $\mathrm{SL}_2(\mathbb{R})^n$

Fix an integer $n \geq 2$. The action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ induces an action of $\mathrm{SL}_2(\mathbb{R})^n$ on $\mathbb{H}^n$, with each matrix acting on the corresponding coordinate. Most of the basic results regarding discrete subgroups of $\mathrm{SL}_2(\mathbb{R})^n$ are easy generalizations of the corresponding results for discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$, but one subject that requires a bit more care is the definition of a cusp for such a subgroup.

We begin by defining what it means for a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})^n$ to have cusp $\infty = (\infty, \ldots, \infty)$: given a discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})^n$, let $\mathbf{t}$ be the set of all vectors $a \in \mathbb{R}^n$ such that the translation $z \mapsto z + a$ is an element of Γ. Since Γ is discrete it follows that $\mathbf{t}$ is a discrete subgroup of $\mathbb{R}^n$, and we will assume that it has rank n.

A vector $\varepsilon \in \mathbb{R}^n$ is said to be totally positive (written $\varepsilon \gg 0$) if $\varepsilon_1 > 0, \ldots, \varepsilon_n > 0$. A totally positive vector ε is said to be a multiplier of Γ if there is some $b \in \mathbb{R}^n$ such that $z \mapsto \varepsilon z + b$ is an element of Γ. One can show that the group Λ of multipliers is a discrete subgroup of $(0, \infty)^n$, and that $\varepsilon_1 \cdots \varepsilon_n = 1$ for all multipliers ε.

The image of Λ under the map $(0, \infty)^n \to \mathbb{R}^n$ given by $(x_1, \ldots, x_n) \mapsto (\log x_1, \ldots, \log x_n)$

is contained in the hyperplane

$$H_0 = \{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\} \tag{3.1}$$

$\log \Lambda$ will be a discrete subgroup of H_0, and order for Γ to have cusp ∞, this subgroup should be a lattice in H_0.

To summarize, Γ is said to have cusp ∞ if $\mathbf{t}$ is a lattice in $\mathbb{R}^n$ and $\log \Lambda$ is a lattice in $H_0 = \{x \in \mathbb{R}^n : x_1 + \cdots + x_n = 0\}$.

This definition then allows us to define cusps in general: a point $\kappa \in (\mathbb{R} \cup \{\infty\})^n$ is said to be a cusp for Γ if $A\Gamma A^{-1}$ has cusp ∞, where A is any element of $\mathrm{SL}_2(\mathbb{R})^n$ such that $A\kappa = \infty$.

Let $(\mathbb{H}^n)^*$ be the union of $\mathbb{H}^n$ and the set of cusps for Γ. Just as for the one-dimensional case, one can define a topology on $(\mathbb{H}^n)^*$ such that the quotient $\Gamma \backslash (\mathbb{H}^n)^*$ becomes a locally compact Hausdorff space. Moreover, the quotient with the (equivalence classes of) the cusps and elliptic points removed is a real manifold of dimension $2n$.

3.2 Hilbert Modular Forms

Let K be a number field, and $\mathcal{O}_K$ the ring of integers of K. If $[K : \mathbb{Q}] = n$ then $K = \mathbb{Q}(\alpha)$ for an algebraic number α having a minimal polynomial of degree n over $\mathbb{Q}$, and therefore there are n embeddings $\sigma : K \to \mathbb{C}$ defined by mapping α to the complex roots of this polynomial. An embedding σ is said to be real if the image of σ is contained in $\mathbb{R}$, and K is totally real if all n embeddings are real.

Let K be a totally real number field of degree n, with embeddings $\sigma_1, \ldots, \sigma_n$. Then the group $\mathrm{SL}_2(K)$ can be embedded in $\mathrm{SL}_2(\mathbb{R})^n$ by the map

$$\gamma \mapsto (\sigma_1(\gamma), \ldots, \sigma_n(\gamma)) \tag{3.2}$$

where $\sigma_j(\gamma)$ means applying σ_j to the entries of γ. As $\mathrm{SL}_2(\mathbb{R})^n$ acts on $\mathbb{H}^n$, this induces an action of $\mathrm{SL}_2(K)$ on $\mathbb{H}^n$.

The Hilbert modular group for K is the subgroup $\Gamma_K = \mathrm{SL}_2(\mathcal{O}_K)$ of $\mathrm{SL}_2(K)$, which we identify with a subgroup of $\mathrm{SL}_2(\mathbb{R})^n$. Γ_K is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})^n$ because $\mathcal{O}_K$ can be identified with a discrete subgroup of $\mathbb{R}^n$.

Moreover, $\mathcal{O}_K$ is in fact a (full) lattice in $\mathbb{R}^n$, and in the notation of the previous section we have $\mathbf{t} = \mathcal{O}_K$. Also $\Lambda = \{\varepsilon^2 : \varepsilon \in \mathcal{O}_K^\times\}$, and it follows from the proof of Dirichlet's units theorem that $\log \Lambda$ is a lattice in H_0. Hence Γ_K has cusp ∞.

In fact, we can say more: the cusps of Γ_K are $K \cup \{\infty\}$. For each $a \in K$ we define a fractional ideal $(a, 1)$, and extend this to ∞ by mapping it to (1). One can show that the induced map from $K \cup \{\infty\}$ to the ideal class group of K is surjective, and that two cusps are Γ_K-equivalent if and only if they define the same ideal class. Hence the number of cusps of Γ_K is equal to the class number of K, so in particular is finite. Moreover, it turns out that the quotient $\Gamma_K \backslash (\mathbb{H}^n \cup K \cup \{\infty\})$ is compact.

With these details out of the way, we can now define modular forms for Γ_K. Fix $r = (r_1, \ldots, r_n) \in \mathbb{Z}^n$, and let $M = (M_1, \ldots, M_n)$ be an element of $\mathrm{SL}_2(\mathbb{R})^n$. If

$$M_j = \begin{bmatrix} a_j & b_j \\ c_j & d_j \end{bmatrix}$$

for each j, then for brevity write

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Also define $N(z) = z_1 \cdots z_n$ for $z = (z_1, \ldots, z_n)$. In analogy with the one-dimensional case, define the factor of automorphy

$$j(M, z) = N(cz + d)^r = \prod_{j=1}^{n} (c_j z_j + d_j)^{r_j} \tag{3.3}$$

This satisfies $j(MN, z) = j(M, Nz) j(N, z)$ for all $M, N \in \mathrm{SL}_2(\mathbb{R})^n$ and all $z \in \mathbb{H}^n$.

A modular form of weight r for Γ_K is a holomorphic function $f : \mathbb{H}^n \to \mathbb{C}$ such that

$$f(Mz) = j(M, z) f(z) \tag{3.4}$$

for all $M \in \Gamma_K$. In the case where all of the components of r are the same integer k (sometimes called parallel weight) we will simply say that f has weight k. As with the one-variable case, there is an additional condition involving the cusps, but unlike the one-variable case, we will see that this condition comes "for free".

3.3 Fourier expansions for Hilbert modular forms

Let us return for the moment to the setting of a general discrete subgroup Γ of $\mathrm{SL}_2(\mathbb{R})^n$. Define $\mathbf{t}$ and Λ as before, and assume that Γ has cusp ∞. Let

$$\mathbf{t}^\vee = \{a \in \mathbb{R}^n : \mathrm{tr}(ax) \in \mathbb{Z} \text{ for all } x \in \mathbf{t}\} \tag{3.5}$$

be the dual lattice of $\mathbf{t}$. In the case $\Gamma = \Gamma_K$ for a totally real number field K, $\mathbf{t} = \mathcal{O}_K$ and $\mathbf{t}^\vee = \mathfrak{d}^{-1}$ is the inverse different ideal.

If f is holomorphic on $\mathbb{H}^n$ and satisfies $f(z+a) = f(z)$ for all $a \in \mathbf{t}$, then f has a Fourier expansion

$$f(z) = \sum_{\nu \in \mathbf{t}^\vee} a_\nu e^{2\pi i \mathrm{tr}(\nu z)} \tag{3.6}$$

with

$$a_\nu = \frac{1}{|P|} \int_P f(z) e^{-2\pi i \mathrm{tr}(\nu z)}\, dx \tag{3.7}$$

where $z = x + iy$, $dx = dx_1 \cdots dx_n$, P is a fundamental polytope for $\mathbf{t}$, and $|P|$ is its (Euclidean) volume. f is said to be regular at ∞ if $a_\nu = 0$ unless $\nu = 0$ or $\nu \gg 0$, and f vanishes at ∞ if in addition $a_0 = 0$.

Next, if κ is any cusp of Γ, choose $A \in \mathrm{SL}_2(\mathbb{R})^n$ such that $A\kappa = \infty$, and define

$$f_A(z) = j(A^{-1}, z)^{-1} f(A^{-1}z) \tag{3.8}$$

One can check that $f_A(Mz) = j(M, z) f_A(z)$ for $M \in A\Gamma A^{-1}$. We then say that f is regular (or vanishes) at κ if f_A is regular (or vanishes) at ∞. While there are many choices of A such that $A\kappa = \infty$, these properties are independent of the choice of A.

We can now complete the definition of a Hilbert modular form for Γ_K of weight r: it is a holomorphic function $f : \mathbb{H}^n \to \mathbb{C}$ such that $f(Mz) = j(M, z) f(z)$ for all $M \in \Gamma_K$, and which is in addition regular at the cusps of Γ_K. If in addition f vanishes at all the cusps then it is said to be a cusp form.

Assuming (as we have throughout this section) that $n \geq 2$, the regularity condition turns out to be superfluous: Koecher's principle states that any holomorphic function $f : \mathbb{H}^n \to \mathbb{C}$ such that $f(Mz) = j(M, z) f(z)$ for all $M \in \Gamma_K$ is necessarily regular at the cusps of Γ_K.

3.4 The L-function of a modular form

Given a modular form $f(z)$ for $\mathrm{SL}_2(\mathbb{Z})$ with Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z} \tag{3.9}$$

define

$$L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}$$

The following lemma implies that $L(s, f)$ converges for $\Re(s) > k$:

Lemma 3.4.1. *Let $f(z)$ be a modular form for $\mathrm{SL}_2(\mathbb{Z})$ with a Fourier expansion as in (3.9). If $f(z)$ is a cusp form, then $a_n = O(n^{\frac{k}{2}})$ as $n \to \infty$, while if f is not a cusp form then $a_n = O(n^{k-1})$ as $n \to \infty$.*

For the proof, first assume that f is a cusp form. Then $|f(z)| = O(e^{-2\pi y})$ as $y \to \infty$ ($z = x + iy$ as usual) since $a_0 = 0$. The function $g(z) = |f(z)|y^{\frac{k}{2}}$ is therefore bounded on the fundamental domain $\mathcal{F}$ for $\mathrm{SL}_2(\mathbb{Z})$, and is also $\mathrm{SL}_2(\mathbb{Z})$-invariant. This implies that

$$|f(z)| \le M y^{-\frac{k}{2}} \tag{3.10}$$

for some constant M and all $z \in \mathbb{H}$.

On the other hand, for any $y > 0$ we have

$$a_n = \int_{iy}^{1+iy} f(z) e^{-2\pi i n z}\, dz \tag{3.11}$$

and using (3.10) we obtain

$$|a_n| \le M y^{-\frac{k}{2}} e^{2\pi i n y} \tag{3.12}$$

and setting $y = \frac{1}{n}$ shows that $|a_n| \le M e^{2\pi} n^{\frac{k}{2}}$.

If $f(z)$ is not a cusp form then we can write $f(z) = g(z) + \alpha E_k(z)$ where $g(z)$ is a cusp form, $\alpha \in \mathbb{C}$, and $E_k(z)$ is the weight k Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$, normalized so that the first Fourier coefficient of E_k is 1. The Fourier coefficients of E_k can be explicitly computed (see Serre [Ser73]) and are $O(n^{k-1})$, and this combined with the growth result just proved for $g(z)$ shows that the Fourier coefficients of f are also $O(n^{k-1})$.

The growth estimate just proved applies specifically to modular forms for $\mathrm{SL}_2(\mathbb{Z})$, but this is just a special case of the general fact that an estimate the growth rate of the Fourier coefficients of f corresponds to an estimate on the growth of $f(z)$ as $\Im(z) \to 0$. More precisely, one can prove the following:

Lemma 3.4.2. *If f is holomorphic on $\mathbb{H}$ with a Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$$

which converges absolutely and locally uniformly, and if there is some $v > 0$ such that $f(z) = O((\Im z)^{-v})$ as $\Im z \to 0$, uniformly in $\Re z$, then it follows that $a_n = O(n^v)$.

Conversely, if $\{a_n\}$ is a sequence of complex numbers with $a_n = O(n^v)$ for some $v > 0$, then the series

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$$

converges absolutely and locally uniformly on $\mathbb{H}$ and defines a holomorphic function $f(z)$ on $\mathbb{H}$ such that $f(z) = O((\Im z)^{-v-1})$ as $\Im z \to 0$ and $f(z) - a_0 = O(e^{-2\pi \Im z})$ as $\Im z \to \infty$, both uniformly in $\Re z$.

Therefore we can safely assume that all of the L-functions below converge in a suitable half-plane. The following result, due to Hecke, shows that modular forms are essentially characterized by the property that their L-functions have functional equations:

Theorem 3.4.1. *Given holomorphic functions*

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z} \qquad g(z) = \sum_{n=0}^{\infty} b_n e^{2\pi i n z}$$

with the two functions satisfying the growth conditions above, and given $N > 0$, define

$$\Lambda_N(s, f) = \left(\frac{2\pi}{\sqrt{N}}\right)^{-s} \Gamma(s) L(s, f) \tag{3.13}$$

for $k > 0$, and similarly for g. Then the following are equivalent:

(i)

$$g(z) = (-i\sqrt{N}z)^{-k} f\Big(-\frac{1}{Nz}\Big) \tag{3.14}$$

and (ii) both $\Lambda_N(s, f)$ *and* $\Lambda_N(s, g)$ *have meromorphic continuations to the entire complex plane, such that*

$$\Lambda_N(s, f) = \Lambda_N(k - s, g) \tag{3.15}$$

and such that

$$\Lambda_N(s, f) + \frac{a_0}{s} + \frac{b_0}{k - s} \tag{3.16}$$

is entire and bounded in any vertical strip.

Taking $f = g$, we obtain the following:

Corollary 3.4.1. *Let* $k \geq 2$ *be an even integer, and let* $f(z)$ *be a holomorphic function on* $\mathbb{H}$ *with a Fourier series satisfying the growth conditions above. Then* f *is a modular form of weight* k *for* $\mathrm{SL}_2(\mathbb{Z})$ *if and only if* $\Lambda(s, f) = (2\pi)^{-s}\Gamma(s)L(s, f)$ *can be meromorphically continued to the entire complex plane and if*

$$\Lambda(s, f) + \frac{a_0}{s} + \frac{(-1)^{\frac{k}{2}} a_0}{k - s} \tag{3.17}$$

is entire and bounded in every vertical strip with $\Lambda(s, f) = (-1)^{\frac{k}{2}}\Lambda(k - s, f)$*. Moreover, if* $a_0 = 0$ *then* f *is a cusp form.*

Hecke's result was generalized by Andre Weil to $\Gamma_0(N)$, but the corresponding result is much more complicated. See Miyake [Miy06] for details.

Finally, if a modular form is an eigenfunction for the Hecke operators, then its L-function has an Euler product. More precisely, we have the following result:

Theorem 3.4.2. *Let* $f(z)$ *be a non-zero modular form of weight* k *for* $\Gamma_0(N)$ *with character* χ *with Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} c_n e^{2\pi i n z}$$

normalized so that $c_1 = 1$*. Then the following are equivalent:*

(i). $f(z)$ is a common eigenfunction of the Hecke operators $T(n)$: for each n there is a complex number $t(n)$ such that $f|T(n) = t(n)f$.

(ii). The L-function of f has an Euler product:

$$L(s,f) = \prod_p (1 - t(p)p^{-s} + \chi(p)p^{k-1-2s})^{-1} \tag{3.18}$$

Moreover, if f satisfies the above conditions then $t(p) = c_p$.

3.5 Base change

We have seen in the previous section that elliptic modular forms are essentially characterized by the properties their L-functions possess. The idea of base change is to associate to an elliptic modular form a corresponding Hilbert modular form in such a way that the L-functions of the two modular forms are related.

More precisely, let K be a totally real number field, and assume for simplicity that K has narrow class number one (so that every ideal is generated by a totally positive element). Then the different ideal is generated by some element d, and so the Fourier expansion for a Hilbert modular form f can be written as

$$f(z) = \sum_{\nu \in \mathcal{O}_K} a_\nu e^{\frac{2\pi i \mathrm{tr}(\nu z)}{d}} \tag{3.19}$$

with $a_\nu = 0$ unless $\nu = 0$ or $\nu \gg 0$.

If ε is a totally positive unit, then there is a unit ε_1 such that $\varepsilon = \varepsilon_1^2$, hence if f is a Hilbert modular form of weight k for $\mathrm{SL}_2(\mathcal{O}_K)$ with k even then

$$f(\varepsilon z) = N(\varepsilon_1)^{-k} f(z) = f(z) \tag{3.20}$$

This implies that $a_{\varepsilon\nu} = a_\nu$ for any totally positive unit, and so the map $\nu \mapsto a_\nu$ can be regarded as a function on ideals of $\mathcal{O}_K$. Therefore we can define the L-function of f to be

$$L(s,f) = \sum_{\mathfrak{a}} a_{\mathfrak{a}} N(\mathfrak{a})^{-s} \tag{3.21}$$

with the sum over integral ideals of $\mathcal{O}_K$. Just as for elliptic modular forms, this L-function has a meromorphic continuation and a functional equation.

Now further suppose that K is a quadratic field of discriminant D. Base change for K takes the following form:

Theorem 3.5.1. *Let $\phi(z) = \sum_n a_n e^{2\pi i n z}$ be a Hecke eigenform of weight k for $\mathrm{SL}_2(\mathbb{Z})$. Then there is a Hilbert modular form f for $\mathrm{SL}_2(\mathcal{O}_K)$ such that*

$$L(s, f) = L(s, \phi) L(s, \phi, \chi_D)$$

Here χ_D is the quadratic character for K, and

$$L(s, \phi, \chi_D) = \sum_n a_n \chi_D(n) n^{-s}$$

is the "twisted" L-function of ϕ.

Base change for general number fields is still an active area of research. The most common approach to base change is via the adelic view of automorphic forms, using the trace formula or converse theorems. However, we will see in chapter four that for the case of real quadratic fields it is possible to construct an explicit kernel function for the Doi-Naganuma lift.

CHAPTER 4

Zagier's Hilbert Modular Forms

This chapter covers the work of Don Zagier in [Zag75], who introduced a family of Hilbert modular forms for quadratic fields which provides an explicit kernel function for the Doi-Naganuma lift.

4.1 Definitions

Let K be a real quadratic field having discriminant $D > 0$, ring of integers $\mathcal{O}_K$, and different ideal $\mathfrak{d}$. Fix an even integer $k \geq 4$, and for each $m \geq 0$ define

$$\omega_m(z_1, z_2) = \sideset{}{'}\sum_{\substack{a,b\in\mathbb{Z},\lambda\in\mathfrak{d}^{-1}\\ \lambda\lambda'-ab=m/D}} (az_1z_2 + \lambda z_1 + \lambda' z_2 + b)^{-k} \tag{4.1}$$

Here $\mathfrak{d}^{-1}$ is the inverse different, $z_1, z_2 \in \mathbb{H}$ and λ' is the conjugate of λ. The prime on the sum indicates that the term $\lambda = a = b = 0$ should be omitted when $m = 0$. $\omega_m(z_1, z_2)$ converges absolutely and locally uniformly on $\mathbb{H} \times \mathbb{H}$, hence is holomorphic. If $m > 0$ then ω_m is a cusp form of weight k for $\mathrm{SL}_2(\mathcal{O}_K)$, and if $m = 0$ then ω_m is a multiple of the Hecke-Eisenstein series of weight k for K.

ω_m can be written in another form: given

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R})$$

define

$$\phi_M(z_1, z_2) = \frac{1}{\det M}\frac{d}{dz_1}\frac{1}{z_2 - Mz_1} = \frac{1}{(cz_1z_2 - az_1 + dz_2 - b)^2} \tag{4.2}$$

for $z_1, z_2 \in \mathbb{H}$. Note that the expression on the right is defined even if $\det M = 0$ (provided that $M \neq 0$), and therefore can be used to define ϕ_M when $\det M = 0$.

If

$$A_j = \begin{bmatrix} \alpha_j & \beta_j \\ \gamma_j & \delta_j \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R}) \qquad (j = 1, 2)$$

then one can check that

$$\phi_M(A_1 z_1, A_2 z_2) = (\gamma_1 z_1 + \delta_1)^2 (\gamma_2 z_2 + \delta_2)^2 \phi_{A_2^* M A_1}(z_1, z_2) \tag{4.3}$$

where $A_2^* = \det(A_2) A_2^{-1}$ is the classical adjoint of A_2. If we set

$$\mathcal{A} = \{M \in M_2(\mathcal{O}_K) : M^* = M'\} \tag{4.4}$$

(where M' is the matrix obtained by conjugating the entries of M) then matrices in $\mathcal{A}$ have the form

$$M = \begin{bmatrix} \theta & b\sqrt{D} \\ -a\sqrt{D} & \theta' \end{bmatrix} \tag{4.5}$$

with $a, b \in \mathbb{Z}$ and $\theta \in \mathcal{O}_K$. Writing $\theta = -\lambda\sqrt{D}$ with $\lambda \in \mathfrak{d}^{-1}$, it follows that

$$\phi_M(z_1, z_2) = D^{-1}(a z_1 z_2 + \lambda z_1 + \lambda' z_2 + b)^{-2} \tag{4.6}$$

and therefore

$$\omega_m(z_1, z_2) = D^{\frac{k}{2}} \sum_{\substack{M \in \mathcal{A}\setminus\{0\} \\ \det M = -m}} \phi_M(z_1, z_2)^{\frac{k}{2}} \tag{4.7}$$

4.2 The Fourier Expansion of ω_m

In this section we will compute the Fourier expansion for ω_m when $m \geq 1$ (as ω_0 is a multiple of the Hecke-Eisenstein series of weight k for K, its Fourier expansion is already known). Before doing so, let us recall what form this expansion takes for a Hilbert modular form:

If $f(z_1, z_2)$ is a holomorphic function on $\mathbb{H} \times \mathbb{H}$ satisfying

$$f(Mz_1, M'z_2) = (\gamma z_1 + \delta)^k (\gamma' z_2 + \delta')^k f(z_1, z_2)$$

for all

$$M = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{SL}_2(\mathcal{O}_K)$$

then in particular we have

$$f(z_1+\theta, z_2+\theta') = f(z_1,z_2)$$

and therefore f has a Fourier expansion

$$f(z_1,z_2) = \sum_{\nu\in\mathfrak{d}^{-1}} a_\nu e^{2\pi i(\nu z_1+\nu' z_2)} \tag{4.8}$$

By Koecher's principle the coefficients $a_\nu = 0$ unless $\nu \gg 0$ (i.e. $\nu > 0$ and $\nu' > 0$), and in order to be a cusp form it is necessary to have $a_0 = 0$ as well (i.e. f vanishes at ∞).

However, this condition is not sufficient if the class number of K is greater than 1; the other cusps must be checked as well. This amounts to the following: for each

$$W = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{SL}_2(K)$$

the function

$$(f|_k W)(z_1,z_2) = (\gamma z_1+\delta)^{-k}(\gamma' z_2+\delta')^{-k} f(Wz_1, W'z_2)$$

must also vanish at infinity.

When $f = \omega_m$, we have

$$(\omega_m|_k W)(z_1,z_2) = D^{\frac{k}{2}}(\gamma z_1+\delta)^{-k}(\gamma' z_2+\delta')^{-k} \sum_{\substack{M\in\mathcal{A}\setminus\{0\} \\ \det M=-m}} \phi_M(Wz_1,W'z_2)^{\frac{k}{2}}$$

$$= D^{\frac{k}{2}} \sum_{\substack{M\in\mathcal{A}\setminus\{0\} \\ \det M=-m}} \phi_{(W')^*MW}(z_1,z_2)^{\frac{k}{2}} = D^{\frac{k}{2}} \sum_{\substack{M\in\mathcal{A}_W\setminus\{0\} \\ \det M=-m}} \phi_M(z_1,z_2)^{\frac{k}{2}}$$

where

$$\mathcal{A}_W = (W')^*\mathcal{A}W$$

As this last expression is so similar to the definition of ω_m, it will be enough to compute the Fourier expansion

$$\omega_m(z_1,z_2) = \sum_{\nu\in\mathfrak{d}^{-1}} c_{m\nu} e^{2\pi i(\nu z_1+\nu' z_2)} \tag{4.9}$$

at infinity.

To that end, begin by writing

$$\omega_m(z_1, z_2) = \omega_m^0(z_1, z_2) + 2\sum_{a=1}^{\infty} \omega_m^a(z_1, z_2) \tag{4.10}$$

where

$$\omega_m^a(z_1, z_2) = \sum_{\substack{b\in\mathbb{Z},\lambda\in\mathfrak{d}^{-1} \\ \lambda\lambda'-ab=m/D}} (az_1z_2 + \lambda z_1 + \lambda' z_2 + b)^{-k} \tag{4.11}$$

Each ω_m^a still satisfies $\omega_m^a(z_1+\theta, z_2+\theta') = \omega_m^a(z_1, z_2)$ for $\theta \in \mathcal{O}_K$, hence has a Fourier expansion

$$\omega_m^a(z_1, z_2) = \sum_{\nu\in\mathfrak{d}^{-1}} c_{m\nu}^a e^{2\pi i(\nu z_1 + \nu' z_2)} \tag{4.12}$$

Therefore

$$c_{m\nu} = c_{m\nu}^0 + 2\sum_{a=1}^{\infty} c_{m\nu}^a \tag{4.13}$$

so it is enough to compute the expansion for each fixed a.

Begin with $a = 0$. In this case the condition $\lambda\lambda' - ab = \frac{m}{D}$ reduces to $\lambda\lambda' = \frac{m}{D}$, and the summation over b is unrestricted. Therefore

$$\omega_m^0(z_1, z_2) = \sum_{\substack{\lambda\in\mathfrak{d}^{-1} \\ \lambda\lambda'=m/D}} \sum_{b\in\mathbb{Z}} (\lambda z_1 + \lambda' z_2 + b)^{-k} \tag{4.14}$$

To compute the Fourier expansion of ω_m^0, we use the "well-known" Fourier expansion

$$\sum_{b\in\mathbb{Z}} (z+b)^{-k} = \frac{(2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1} e^{2\pi i r z} \tag{4.15}$$

Moreover, if $\lambda\lambda' = \frac{m}{D} > 0$, either $\lambda \gg 0$ or $-\lambda \gg 0$, and since k is even we can replace λ with $-\lambda$ if $-\lambda \gg 0$.

Therefore setting $z = \lambda z_1 + \lambda' z_2$, we obtain

$$\omega_m^0(z_1, z_2) = 2\frac{(2\pi i)^k}{(k-1)!} \sum_{\substack{\lambda\in\mathfrak{d}^{-1},\lambda\gg 0 \\ \lambda\lambda'=m/D}} \sum_{r=1}^{\infty} r^{k-1} e^{2\pi i r(\lambda z_1 + \lambda' z_2)} \tag{4.16}$$

Hence $c_{m\nu}^0 = 0$ unless $\nu \gg 0$ and $\nu = r\lambda$ for some $r \geq 1$ and some $\lambda \in \mathfrak{d}^{-1}$ with $\lambda\lambda' = \frac{m}{D}$, in which case

$$c_{m\nu}^0 = 2\frac{(2\pi i)^k}{(k-1)!} r^{k-1} \tag{4.17}$$

For $a > 0$ things are considerably more complicated. We begin by writing

$$\omega_m^a(z_1, z_2) = \sum_{M \in S} \phi_M(z_1, z_2)^{\frac{k}{2}} \tag{4.18}$$

where

$$S = \left\{ \begin{bmatrix} -\lambda & -b \\ a & \lambda' \end{bmatrix} : \lambda \in \mathfrak{d}^{-1}, b \in \mathbb{Z}, \lambda\lambda' - ab = \frac{m}{D} \right\} \tag{4.19}$$

If $M \in S$ and $\theta \in \mathcal{O}_K$, define

$$M_\theta = \begin{bmatrix} 1 & \theta' \\ 0 & 1 \end{bmatrix}^{-1} M \begin{bmatrix} 1 & \theta \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -\lambda - a\theta' & -b - \theta\lambda - \theta'\lambda' - a\theta\theta' \\ a & \lambda' + a\theta \end{bmatrix} \in S$$

Conversely, if

$$M_1 = \begin{bmatrix} -\lambda_1 & -b_1 \\ a & \lambda_1' \end{bmatrix} \in S$$

with $\lambda_1 = \lambda + a\theta'$, then

$$b_1 = \frac{\lambda_1\lambda_1' - \frac{m}{D}}{a} = b + \theta\lambda + \theta'\lambda' + a\theta\theta'$$

so $M_1 = M_\theta$. Therefore

$$\omega_m^a(z_1, z_2) = \sum_{M \in S} \phi_M(z_1, z_2)^{\frac{k}{2}} = \sum_{\lambda \in R} \sum_{\theta \in \mathcal{O}_K} \phi_{M(\lambda)_\theta}(z_1, z_2)^{\frac{k}{2}} \tag{4.20}$$

where R is a set of representatives for the set of $\lambda \in \mathfrak{d}^{-1}/a\mathcal{O}_K$ for which $N(\lambda\sqrt{D}) \equiv -m \pmod{aD}$ and

$$M(\lambda) = \begin{bmatrix} -\lambda & -b \\ a & \lambda' \end{bmatrix} \qquad b = \frac{\lambda\lambda' - \frac{m}{D}}{a} \tag{4.21}$$

Since $\phi_M(z_1 + \theta, z_2 + \theta') = \phi_{M_\theta}(z_1, z_2)$, each of the inner sums above has a Fourier expansion, and since R is finite, this will determine the Fourier expansion of ω_m^a.

To compute the Fourier expansion of the inner sum, begin with the following result:

Lemma 4.2.1. *If* $\alpha > 0$ *then*

$$\sum_{\theta \in \mathcal{O}_K} [(z_1 + \theta)(z_2 + \theta') - \alpha)]^{-k} = \sum_{\nu \in \mathfrak{d}^{-1}} j_{k-1}(\alpha, \nu) e^{2\pi i(\nu z_1 + \nu' z_2)}$$

with

$$j_{k-1}(\alpha,\nu) = \frac{(2\pi)^{k+1}}{(k-1)!\sqrt{D}}\Big(\frac{N(\nu)}{\alpha}\Big)^{\frac{k-1}{2}} J_{k-1}(4\pi\sqrt{\alpha N(\nu)}) \tag{4.22}$$

for $J_{k-1}(z)$ *a Bessel function.*

The proof is via a contour integration: as the sum on the left is invariant under the maps $T_\theta : (z_1, z_2) \to (z_1+\theta, z_2+\theta')$, it has a Fourier expansion as on the right with

$$j_{k-1}(\alpha,\nu) = \frac{1}{\sqrt{D}}\int_A \sum_{\theta\in\mathcal{O}_K} [(z_1+\theta)(z_2+\theta')-\alpha)]^{-k} e^{-2\pi i(\nu z_1+\nu' z_2)}\, dz_1 dz_2 \tag{4.23}$$

with A a fundamental domain for the action of the maps T_θ on the plane $\mathcal{P} = \{\Im z_1 = c_1, \Im z_2 = c_2\}$ (A has area $\sqrt{D}$, hence the normalizing factor in j_{k-1}).

As A is compact, we can interchange the sum and integral, and as the domains $T_\theta A$ cover $\mathcal{P}$, the result is

$$j_{k-1}(\alpha,\nu) = \frac{1}{\sqrt{D}}\int_{\Im z_2=c_2}\int_{\Im z_1=c_1} (z_1 z_2-\alpha)^{-k} e^{-2\pi i(\nu z_1+\nu' z_2)}\, dz_1 dz_2$$

$$= \frac{1}{\sqrt{D}}\int_{\Im z_2=c_2} z_2^{-k} e^{-2\pi i\nu' z_2}\int_{\Im z_1=c_1}\Big(z_1-\frac{\alpha}{z_2}\Big)^{-k} e^{-2\pi i\nu z_1}\, dz_1 dz_2$$

The integrand in the inner integral has a pole at $z_1 = \frac{\alpha}{z_2}$, which is in the lower half plane since $\Im z_2 > 0$. Therefore if $\nu \le 0$ then we can shift contours upward to $i\infty$ to get $j_{k-1}(\alpha,\nu) = 0$. The same argument works if $\nu' \le 0$, and so $j_{k-1}(\alpha,\nu) = 0$ unless $\nu \gg 0$.

In this case we shift contours downward, and the inner integral is

$$-2\pi i\operatorname{res}_{z_1=\frac{\alpha}{z_2}}\Big(\Big(z_1-\frac{\alpha}{z_2}\Big)^{-k} e^{-2\pi i(\nu z_1+\nu' z_2)}\Big) = \frac{(-2\pi i)^k\nu^{k-1}}{(k-1)!} e^{-2\pi i\nu\frac{\alpha}{z_2}}$$

hence

$$j_{k-1}(\alpha,\nu) = \frac{(-2\pi i)^k\nu^{k-1}}{(k-1)!\sqrt{D}}\int_{\Im z_2=c_2} z_2^{-k} e^{-2\pi i\nu' z_2-2\pi i\nu\frac{\alpha}{z_2}}\, dz_2 \tag{4.24}$$

and making the change of variables $z_2 = i\sqrt{\frac{\alpha\nu}{\nu'}}t$, we get

$$j_{k-1}(\alpha,\nu) = -i\frac{(2\pi)^k}{(k-1)!\sqrt{D}}\Big(\frac{N(\nu)}{\alpha}\Big)^{\frac{k-1}{2}}\int_{c_2'-i\infty}^{c_2'+i\infty} t^{-k} e^{2\pi\sqrt{\alpha\nu\nu'}(t-\frac{1}{t})}\, dt \tag{4.25}$$

As the integral on the right-hand side of (4.25) is

$$2\pi i J_{k-1}(4\pi\sqrt{\alpha\nu\nu'}) \tag{4.26}$$

this completes the computation of j_{k-1}.

Now suppose that

$$M = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

with $\det M = -\Delta < 0$. Then

$$\phi_M(z_1, z_2) = (\gamma z_1 z_2 - \alpha z_2 + \delta z_2 - \beta)^{-2} = \gamma^{-2}[(z_1 + \delta/\gamma)(z_2 - \alpha/\gamma) - \Delta/\gamma^2]^{-2}$$

and so using the result above we obtain

$$\sum_{\theta \in \mathcal{O}_K} \phi_M(z_1, z_2) = \sum_{\nu \in \mathfrak{d}^{-1}} \gamma^{-k} e^{2\pi i(\frac{\nu\delta}{\gamma} - \frac{\nu'\alpha}{\gamma})} j_{k-1}\Big(\frac{\Delta}{\gamma^2}, \nu\Big) e^{2\pi i(\nu z_1 + \nu' z_2)} \tag{4.27}$$

Finally, setting $\alpha = -\lambda, \beta = -b, \gamma = a, \delta = \lambda'$, we get

$$c^a_{m\nu} = \frac{(2\pi)^{k+1}}{(k-1)!} \frac{D^{\frac{k}{2}-1}}{a} \Big(\frac{N(\nu)}{m}\Big)^{\frac{k-1}{2}} G_a(m, \nu) J_{k-1}\Big(\frac{4\pi}{a}\sqrt{\frac{mN(\nu)}{D}}\Big) \tag{4.28}$$

with

$$G_a(m, \nu) = \sum_{\substack{\lambda \in \mathfrak{d}^{-1}/a\mathcal{O}_K \\ \lambda\lambda' \equiv m/D (a\mathbb{Z})}} e^{2\pi i \frac{\mathrm{tr}(\nu\lambda)}{a}} \tag{4.29}$$

4.3 Poincaré series for $\Gamma_0(D)$

From this point on, suppose for simplicity that $D \equiv 1 \pmod 4$, so that (since D is the discriminant of a quadratic field) D is square-free. Then one can check that two points $\frac{x}{y}, \frac{x'}{y'}$ of $\mathbb{Q} \cup \{\infty\}$ are $\Gamma_0(D)$-equivalent if and only if $\gcd(y, D) = \gcd(y', D)$, hence the equivalence classes of $\mathbb{Q} \cup \{\infty\}$ modulo $\Gamma_0(D)$ correspond to positive divisors D_1 of D.

Let $P = D_1$ be a cusp of $\Gamma_0(D)$, and set $D_2 = \frac{D}{D_1}$. Then D_1 and D_2 are coprime since D is square-free, hence there are integers p, q such that $pD_1 + qD_2 = 1$, and we can choose

$$A_P = \begin{bmatrix} D_2 & -p \\ D_1 & q \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

The width of the cusp P is D_2.

Let ε be the fundamental quadratic character of K, defined by $\varepsilon(p) = \left(\frac{D}{p}\right)$ for primes p not dividing $2D$, and extend ε to $\Gamma_0(D)$ by

$$\varepsilon\Big(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\Big) = \varepsilon(a) = \varepsilon(d) \tag{4.30}$$

Let $S(D, k, \varepsilon)$ be the space of cusp forms of weight k for $\Gamma_0(D)$ and ε. Writing D_1 for P, any $f \in S(D, k, \varepsilon)$ has a Fourier expansion of the form

$$(f|_k A_{D_1}^{-1})(z) \sum_{n=1}^{\infty} a_n^{D_1}(f) e^{\frac{2\pi i n z}{D_2}} \tag{4.31}$$

with

$$a_n^{D_1}(f) = \frac{(4\pi n)^{k-1}}{(k-2)!} D_2^{-k}(f, G_n^{D_1}) \tag{4.32}$$

The Fourier expansions for the Poincare series $G_n^{D_1}$ now take the form

$$G_n^{D_1}(z) = \sum_{m=1}^{\infty} g_{nm}^{D_1} e^{2\pi i m z} \tag{4.33}$$

with

$$g_{nm}^{D_1} = \delta_{D_1 D}\delta_{nm} + 2\pi i^k \Big(\frac{mD_2}{n}\Big)^{\frac{k-1}{2}} \sum_{\substack{c=1 \\ (c,D)=D_1}}^{\infty} H_c^{D_1}(n, m) J_{k-1}\Big(\frac{4\pi}{c}\sqrt{\frac{mn}{D_2}}\Big) \tag{4.34}$$

and

$$H_c^{D_1}(n, m) = \frac{1}{c} \sum_{\substack{d \mod c \\ (d,c)=1}} \varepsilon\Big(A_P^{-1}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\Big) e^{\frac{2\pi i}{c}(\frac{na}{D_2}+md)} \tag{4.35}$$

This last sum can be simplified a bit: since D_1 divides c, $A_P^{-1}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ lies in $\Gamma_0(D)$ precisely when $D_2|a$. Hence a is determined mod cD_2 by $D_2|a$ and $ad \equiv 1 \pmod{c}$, and

$$\varepsilon\Big(A_P^{-1}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\Big) = \varepsilon(ap + pc) = \Big(\frac{aq + pc}{D}\Big) = \Big(\frac{aq + pc}{D_1}\Big)\Big(\frac{aq + pc}{D_2}\Big)$$

$$= \left(\frac{aq}{D_1}\right)\left(\frac{pc}{D_2}\right) = \left(\frac{a/D_2}{D_1}\right)\left(\frac{c/D_1}{D_2}\right) = \left(\frac{-d}{D_1}\right)\left(\frac{c}{D_2}\right)$$

with the last equality following from $\left(\frac{D_2}{D_1}\right)\left(\frac{D_1}{D_2}\right) = \left(\frac{-1}{D_1}\right)$ by quadratic reciprocity. Therefore

$$H_c^{D_1}(n,m) = \frac{1}{c}\left(\frac{c}{D_2}\right) \sum_{\substack{d \mod c \\ (d,c)=1}} \left(\frac{-d_1}{D_2}\right) e^{\frac{2\pi i}{c}\left(\frac{n}{dD_2}+md\right)} \tag{4.36}$$

Next, we need the following linear combinations of Poincare series:

$$G_n(z) = \sum_{\substack{D=D_1D_2 \\ D_2|n}} \psi(D_2) D_2^{-k} G_{n/D_2}^{D_1}(z) \tag{4.37}$$

where $\psi(D_2) = \left(\frac{D_1}{D_2}\right)\sqrt{D_2}$ if $D_1 \equiv D_2 \equiv 1 \pmod 4$, and $\psi(D_2) = -i\left(\frac{D_1}{D_2}\right)\sqrt{D_2}$ if $D_1 \equiv D_2 \equiv 3 \pmod 4$. Note that ψ is just a Gauss sum. Also, if $D_2 = D_2' D_2''$ divides D, then

$$\psi(D_2) = \psi(D_2')\psi(D_2'')$$

Using the Fourier expansions of the Poincare series $G_{n/D_2}^{D_1}$, we get

$$G_n(z) = \sum_{m=1}^{\infty} g_{nm} e^{2\pi i m z} \tag{4.38}$$

with

$$g_{nm} = \sum_{\substack{D=D_1D_2 \\ D_2|n}} \psi(D_2) D_2^{-k} g_{n/D_2 m}^{D_1}$$

$$= \delta_{nm} + 2\pi i^k \left(\frac{m}{n}\right)^{\frac{k-1}{2}} \sum_{b=1}^{\infty} H_b(n,m) J_{k-1}\left(\frac{4\pi}{bD}\sqrt{nm}\right) \tag{4.39}$$

and

$$H_b(n,m) = \sum_{\substack{D=D_1D_2 \\ D_2|n \\ (b,D_2)=1}} \frac{\psi(D_2)}{D_2} H_{bD_1}^{D_1}\left(\frac{n}{D_2}, m\right) \tag{4.40}$$

4.4 The functions $\Omega(z_1, z_2; \tau)$

Let D be a positive squarefree integer, $D \equiv 1 \pmod 4$, and $k \geq 4$ an even integer. For $z_1, z_2, \tau \in \mathbb{H}$, define

$$\Omega(z_1, z_2; \tau) = \sum_{m=1}^{\infty} m^{k-1} \omega_m(z_1, z_2) e^{2\pi i m\tau} \tag{4.41}$$

This series converges absolutely due to the exponential decay of the terms $e^{2\pi i m\tau}$. For each fixed τ, $\Omega(z_1, z_2; \tau)$ is a Hilbert modular form for $\mathrm{SL}_2(\mathcal{O}_K)$. In this section, we will show that if z_1, z_2 are fixed then Ω is a cusp form for $\Gamma_0(D)$ with character ε. The key result is the following:

Theorem 4.4.1. *For all $z_1, z_2, \tau \in \mathbb{H}$,*

$$\Omega(z_1, z_2; \tau) = \sum_{n=1}^{\infty} n^{k-1} \omega_n^0(z_1, z_2) G_n(\tau) \tag{4.42}$$

where ω_m^0 is the sum defined above when computing the Fourier expansion of ω_m.

The proof consists of comparing two Fourier series, and will be carried out through a series of reductions, which we will state as lemmas. On the one hand, we have already found that

$$\Omega(z_1, z_2; \tau) = \sum_{m=1}^{\infty} \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} m^{k-1} c_{m\nu} e^{2\pi i m\tau} e^{2\pi i(\nu z_1 + \nu' z_2)}$$

with the coefficients $c_{m\nu}$ computed in section 4.2.

On the other hand, we have also computed

$$\omega_n^0(z_{1,2}) = 2\frac{(2\pi i)^k}{(k-1)!} \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} \Big[\sum_{\substack{r|\nu \\ D\nu\nu' = nr^2}} r^{k-1} \Big] e^{2\pi i(\nu z_1 + \nu' z_2)}$$

where the inner sum is over natural numbers r such that $\frac{\nu}{r} \in \mathfrak{d}^{-1}$ and $N(\frac{\nu}{r}) = \frac{n}{D}$, and in fact contains at most one term.

Using the Fourier of expansion of $G_n(\tau)$, the right-hand side of (4.42) therefore becomes

$$\sum_{m=1}^{\infty} \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} \Big(2\frac{(2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} g_{nm} \sum_{\substack{r|\nu \\ D\nu\nu' = nr^2}} r^{k-1} \Big) e^{2\pi i(\nu z_1 + \nu' z_2)} e^{2\pi i m\tau}$$

$$= 2\Big(2\frac{(2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty}\sum_{\substack{\nu\in\mathfrak{d}^{-1}\\ \nu\gg 0}}\Big(\sum_{r|\nu}\Big(\frac{D\nu\nu'}{r}\Big)^{k-1} g_{\frac{D\nu\nu'}{r^2},m}\Big)e^{2\pi i(\nu z_1+\nu' z_2)}e^{2\pi i m\tau}$$

To complete the proof, it is therefore enough to show that

$$m^{k-1}c_{m\nu} = 2\frac{(2\pi i)^k}{(k-1)!}\sum_{r|\nu}\Big(\frac{D\nu\nu'}{r}\Big)^{k-1} g_{\frac{D\nu\nu'}{r^2},m} \tag{4.43}$$

Using the definitions of $c_{m\nu}$ and g_{nm}, this reduces to establishing the identity

$$\frac{1}{a\sqrt{D}}G_a(m,\nu) = \sum_{r|\nu, r|a} H_{\frac{a}{r}}\Big(\frac{D\nu\nu'}{r^2}, m\Big) \tag{4.44}$$

for integers $a, m \in \mathbb{Z}$ with $a > 0$ and $\nu \in \mathfrak{d}^{-1}$.

To prove this identity, let $\mu = \sqrt{D}\nu \in \mathcal{O}_K$, so that

$$G_a(m,\nu) = \sum_{\substack{\lambda \mod a\mathfrak{d} \\ N(\lambda)=-m \mod aD}} e^{\frac{2\pi i \mathrm{tr}(\lambda\mu)}{aD}} \tag{4.45}$$

We want to show that this sum is equal to

$$a\sqrt{D}\sum_{r|\mu, r|a} H_{\frac{a}{r}}\Big(-N\Big(\frac{\mu}{r}\Big), m\Big) = a\sqrt{D}\sum_{r|\mu, r|a}\sum_{\substack{D=D_1D_2\\ D_2|N(\mu/r)\\ (a/r,D_2)=1}}\frac{\psi(D_2)}{D_2}H^{D_1}_{\frac{a}{r}D_1}(-N(\mu)/r^2D_2, m) \tag{4.46}$$

As both expressions in (4.46) are periodic in m with period aD, it is enough to show that their finite Fourier transforms coincide. Hence it is enough to prove the following:

Lemma 4.4.1.

$$\sum_{m=1}^{aD} e^{-\frac{2\pi i h m}{aD}} \sum_{\substack{\lambda \mod a\mathfrak{d} \\ N(\lambda)=-m \mod aD}} e^{\frac{2\pi i \mathrm{tr}(\lambda\mu)}{aD}}$$

$$= a\sqrt{D}\sum_{r}\sum_{D_2}\frac{\psi(D_2)}{D_2}\sum_{m=1}^{aD} e^{-\frac{2\pi i h m}{aD}} H^{D_1}_{\frac{a}{r}D_1}(-N(\mu)/r^2D_2, m)$$

for all $h \in \mathbb{Z}/aD\mathbb{Z}$.

The first expression in lemma 4.4.1 simplifies to

$$\sum_{\lambda \mod a\mathfrak{d}} e^{\frac{2\pi i h N(\lambda)+\mathrm{tr}(\mu\lambda)}{aD}} \tag{4.47}$$

For the second, note that $H_c^{D_1}(n,m)$ is a linear combination of terms ζ^m with ζ a primitive cth root of unity. Therefore if $c|aD$ then the sum

$$\sum_{m \mod aD} e^{-\frac{2\pi ihm}{aD}} H_c^{D_1}(n,m)$$

is zero unless $(h,aD) = \frac{aD}{c}$, in which case it is equal to

$$\frac{aD}{c}\Big(\frac{c}{D_2}\Big)\Big(\frac{-d}{D_1}\Big)e^{\frac{2\pi in}{cD_2d}} \tag{4.48}$$

with d defined by $\frac{h}{aD} = \frac{d}{c}$. Hence the inner sum in the second expression is zero unless $(h,aD) = rD_2$. On the other hand, the conditions on r and D_2 imply that $(a,rD_2) = r$, so $(a,h) = (a,h,aD) = (a,rD_2) = r$. Thus r and D_2 are determined by h:

$$r = (a,h) \qquad D_2 = \frac{(h,aD)}{(h,a)} \tag{4.49}$$

This reduces the desired identity in lemma 4.4.1 to another one:

Lemma 4.4.2. *With r, D_1, D_2 as above,*

$$\sum_{\lambda\in\mathcal{O}_K/a\mathfrak{d}} e^{\frac{2\pi i(h\lambda\lambda'+\mu\lambda+\mu'\lambda')}{aD}} = a\sqrt{D}\frac{\psi(D_2)}{D_2}rD_2\Big(\frac{aD_1/r}{D_2}\Big)\Big(\frac{-h/rD_2}{D_1}\Big)e^{-\frac{2\pi i(\frac{\mu\mu'}{r^2D_2}\frac{r}{h})}{aD_1/r}} \tag{4.50}$$

if $r|\mu, D_2|N(\frac{\mu}{r})$, and is zero otherwise.

First let us show that the sum is zero unless $r|\mu$ and $D_2|N(\frac{\mu}{r})$. Replacing λ with $\lambda+\frac{a}{r}\tau\sqrt{D}$ in the sum (with τ an algebraic integer), then $hN(\lambda)$ becomes

$$hN\Big(\lambda+\frac{a}{r}\tau\sqrt{D}\Big) = hN(\lambda) + aD\frac{h}{r}\mathrm{tr}\Big(\frac{\lambda'\tau}{\sqrt{D}}\Big) - aD\frac{ah}{r^2}N(\tau) \equiv hN(\lambda) \mod aD$$

and $\mathrm{tr}(\mu\lambda)$ becomes

$$\mathrm{tr}(\mu\lambda) + \frac{aD}{r}\mathrm{tr}\Big(\frac{\mu\tau}{\sqrt{D}}\Big)$$

Therefore the effect of replacing λ with $\lambda+\frac{a}{r}\tau\sqrt{D}$ is to multiply the summand by a factor

$$e^{\frac{2\pi i\mathrm{tr}(\frac{\mu\tau}{\sqrt{D}})}{r}}$$

independent of λ. Hence the sum is non-zero only if this factor is 1 for all τ, i.e. only if $\mathrm{tr}(\frac{\mu\tau}{\sqrt{D}})$ is an integer for all τ. And this is the case precisely when $\frac{\mu}{r\sqrt{D}} \in \mathfrak{d}^{-1}, r|\mu$.

Next, we can apply the same argument with $\tau \in \mathfrak{a}^{-1}$, where $\mathfrak{a}$ is the ideal defined by $\mathfrak{a}^2 = (D_2)$. Since D_2 divides $\frac{h}{r}$, we can again conclude that replacing λ with $\lambda + \frac{a}{r}\tau\sqrt{D}$ does not change $hN(\lambda)$ mod aD, and so

$$e^{\frac{2\pi i \mathrm{tr}(\frac{\mu\tau}{\sqrt{D}})}{r}} = 1$$

for all $\tau \in \mathfrak{a}^{-1}$. Hence we must have

$$\frac{\mu}{r\sqrt{D}} \in \mathfrak{a}\mathfrak{d}^{-1} \qquad \frac{\mu}{r} \in \mathfrak{a} \qquad D_2 | N\Big(\frac{\mu}{r}\Big)$$

Now assume that $r|\mu$ and $D_2|N(\frac{\mu}{r})$. This means that a, h, and μ are all divisible by r, and replacing a, h, μ and r in the identity above with $\frac{a}{r}, \frac{h}{r}, \frac{\mu}{r}$, and 1 has the effect of dividing both sides by r^2. So we may assume that $r = 1$.

In this case $\gcd(h, a) = 1$ and $D_2 = \gcd(h, D)$ divides $\mu\mu'$, and the desired identity in lemma 4.4.2 reduces to proving

Lemma 4.4.3.

$$\sum_{\lambda \in \mathcal{O}_K/a\mathfrak{d}} e^{\frac{2\pi i(h\lambda\lambda' + \mu\lambda + \mu'\lambda')}{aD}} = a\sqrt{D}\psi(D_2)\Big(\frac{aD_1}{D_2}\Big)\Big(\frac{-h/D_2}{D_1}\Big)e^{-\frac{2\pi i \frac{N(\mu)}{D_2} h^{-1}}{aD_1}} \tag{4.51}$$

Since $D_2|h$ and $D_2|\mu\mu'$, it follows that

$$e^{\frac{2\pi i(h\lambda\lambda' + \mu\lambda + \mu'\lambda')}{aD}} = e^{\frac{2\pi i \frac{h}{D_2} N(\lambda + h^{-1}\mu')}{aD_1}} e^{-\frac{2\pi i h^{-1} \frac{N(\mu)}{D_2}}{aD_1}}$$

with $\frac{h}{D_2} \in \mathbb{Z}$ and $\frac{\mu}{D_2} \in \mathfrak{d}^{-1}$. So multiplying both sides by $e^{\frac{2\pi i h^{-1}\frac{N(\mu)}{D_2}}{aD_1}}$ and replacing λ with $\lambda - h^{-1}\mu'$ (mod aD_1), h with $\frac{h}{D_2}$, and aD_1 with b, the identity of lemma 4.4.3 reduces to

$$\sum_{\lambda \in \mathcal{O}_K/a\mathfrak{d}} e^{\frac{2\pi i h\lambda\lambda'}{b}} = a\sqrt{D}\psi(D_2)\Big(\frac{b}{D_2}\Big)\Big(\frac{-h}{D_1}\Big) \tag{4.52}$$

for $(h, b) = 1, (b, D) = D_1, \frac{D}{D_1} = D_2, \frac{b}{D_1} = a$. If we replace the sum of $\mathcal{O}_K/a\mathfrak{d}$ with a sum over $\mathcal{O}_K/aD_1$, the effect is to multiply its value by $\frac{D_1}{D_2}$, since both sums are multiples of a sum over $\mathcal{O}_K/a\mathfrak{a}$ with $\mathfrak{a}^2 = (D_1)$. Hence we have now reduced the problem to proving the following:

Lemma 4.4.4.

$$\frac{1}{b}\sum_{\lambda\in\mathcal{O}_K/b} e^{\frac{2\pi i h N(\lambda)}{b}} = \xi\Big(\frac{b/D_1}{D_2}\Big)\Big(\frac{h}{D_1}\Big)\sqrt{D_1} \tag{4.53}$$

where $\xi = 1$ if $D_1 \equiv 1$ (mod 4) and $\xi = i$ if $D_1 \equiv 3$ (mod 4).

Writing $C(h/b)$ for the left-hand side of (4.53), we see that $C(h/b)$ depends on the class of h in $(\mathbb{Z}/b\mathbb{Z})^\times$ mod the squares. Also, if $b = b_1 b_2$ with $(b_1, b_2) = 1$, then

$$C\Big(\frac{h}{b}\Big) = C\Big(\frac{hb_2}{b_1}\Big)C\Big(\frac{hb_1}{b_2}\Big)$$

The same is true for the right-hand side of (4.53) by the properties of Kronecker symbols, hence it is enough to consider the case when $b = q^t$ is a prime power. Now write

$$C\Big(\frac{h}{b}\Big) = \frac{1}{b}\sum_{n=1}^{b} N_b(n) e^{\frac{2\pi i h n}{b}} \tag{4.54}$$

with

$$N_b(n) = \#\{\lambda \in \mathcal{O}_K/b : N(\lambda) \equiv n \mod b\}$$

To evaluate $C\Big(\frac{h}{b}\Big)$ in the case $b = q^t$, it is therefore enough to determine $N_b(n)$. Write $n = n_0 q^v$ with $(q, n_0) = 1$. There turn out to be three cases, which are tedious but straightforward to verify:

Lemma 4.4.5. *(i). If $(\frac{D}{q}) = 1$, then $N_b(n) = (v+1)q^{t-1}(q-1)$ if $v < t$, and $N_b(n) = (t+1)q^t - tq^{t-1}$ if $v \geq t$.*

(ii). If $(\frac{D}{q}) = -1$, then $N_b(n) = q^{t-1}(q+1)$ if $v < t$ with v even, $N_b(n) = 0$ if $v < t$ with v odd, $N_b(n) = q^t$ if $v \geq t$ with t even, and $N_b(n) = q^{t-1}$ if $v \geq t$ with t odd.

(iii). If q divides D and $D_2 = \frac{D}{q}$, then $N_b(n) = (1 + (\frac{n_0}{q}))q^t$ if $v < t$ with v even, $N_b(n) = (1 + (\frac{-n_0 D_2}{q}))q^t$ if $v < t$ with v odd, and $N_b(n) = q^t$ if $v \geq t$.

Using lemma 4.4.5, first suppose that $b = q^t$ with q not dividing D. Then

$$C\Big(\frac{h}{b}\Big) = \frac{1}{b}\sum_{v=0}^{t} N_b(q^v) \sum_{\substack{n=1\\ q^v \| n}}^{b} e^{\frac{2\pi i h n}{b}} \tag{4.55}$$

and the inner sum is 1 if $v=t$, -1 if $v=t-1$, and 0 if $v<t-1$. Therefore in this case

$$C\Big(\frac{h}{b}\Big)=\frac{1}{b}\Big[N_b(q^t)-N_b(q^{t-1})\Big] \tag{4.56}$$

If $(\frac{D}{q})=1$, then (4.56) is equal to 1, which agrees with the right-hand side of (4.53), since in this case $D_1=1, D_2=D$, and $(\frac{b}{D_2})=1$.

If $(\frac{D}{q})=-1$ then (4.56) is $(-1)^t=(\frac{b}{D})$, which also agrees with the right-hand side of (4.53).

Finally, if $q|D$ and $D_2=\frac{D}{q}$,

$$C\Big(\frac{h}{b}\Big)=\sum_{\substack{0\le v<t\\ v \text{ even}}}\ \sum_{\substack{n_0 \bmod q^{t-v}\\ (n_0,q)=1}}\Big(\frac{n_0}{q}\Big)e^{\frac{2\pi i h n_0}{q^{t-v}}}+\Big(\frac{-D_2}{q}\Big)\sum_{\substack{0\le v<t\\ v \text{ odd}}}\ \sum_{\substack{n_0 \bmod q^{t-v}\\ (n_0,q)=1}}\Big(\frac{n_0}{q}\Big)e^{\frac{2\pi i h n_0}{q^{t-v}}} \tag{4.57}$$

Replacing n_0 with n_0+q, we see that both inner sums are zero if $v<t-1$. Hence

$$C\Big(\frac{h}{b}\Big)=\Big(\frac{h}{q}\Big)\xi\sqrt{q} \tag{4.58}$$

if t is odd, and

$$C\Big(\frac{h}{b}\Big)=\Big(\frac{-D_2}{q}\Big)\Big(\frac{h}{q}\Big)\xi\sqrt{q} \tag{4.59}$$

if t is even. Once again $\xi=1$ if $q\equiv 1 \pmod 4$, and $\xi=i$ if $q\equiv 3 \pmod 4$. As $D_1=q$ and $(\frac{-D_2}{q})=(\frac{q}{D_2})$, we can rewrite $C(\frac{h}{b})$ as

$$C\Big(\frac{h}{b}\Big)=\Big(\frac{q}{D_2}\Big)^{t-1}\Big(\frac{h}{q}\Big)\varepsilon\sqrt{q}=\Big(\frac{b/D_1}{D_2}\Big)\Big(\frac{h}{D_1}\Big)\varepsilon\sqrt{D_1} \tag{4.60}$$

This agrees with the right-hand side of (4.53), hence completes the proof of the theorem.

4.5 The Doi-Naganuma map

We saw in the previous section that

$$\Omega(z_1,z_2;\tau)=\sum_{m=1}^{\infty}m^{k-1}\omega_m(z_1,z_2)e^{2\pi i m\tau}=\sum_{m=1}^{\infty}m^{k-1}\omega_m^0(z_1,z_2)G_m(\tau)$$

In particular, for each fixed $\tau\in\mathbb{H}$, the series on the right defines a cusp form of weight k for $\mathrm{SL}_2(\mathcal{O}_K)$. In this section, we will investigate the consequences of this fact. Define

$S_1 = S_k(\Gamma_0(D), \varepsilon)$ where ε is the quadratic character of K, and let S_2 be the space of cusp forms of weight k for $\mathrm{SL}_2(\mathcal{O}_K)$.

Then Ω defines a map $S_1 \to S_2$ via the Petersson inner product:

$$f \mapsto (f, \Omega)_\tau = \int_{\mathcal{F}} f(\tau)\overline{\Omega(z_1, z_2; \tau)} y^{k-2}\, dxdy$$

where $\mathcal{F}$ is a fundamental domain for $\Gamma_0(D)$.

However, in the previous section we expressed Ω as a sum of Poincare series, which allows us to evaluate the Petersson inner product of Ω with any cusp form. If we write $a_n^{D_1}(f)$ for the Fourier coefficients of f at the cusps of $\Gamma_0(D)$, then

$$\begin{aligned} n^{k-1}(f, G_n) &= n^{k-1} \sum_{\substack{D = D_1 D_2 \\ D_2 | n}} \psi(D_2) D_2^{-k} (G^{D_1}_{n/D_2}, f) \\ &= \frac{(k-2)!}{(4\pi)^{k-1}} \sum_{\substack{D = D_1 D_2 \\ D_2 | n}} \psi(D_2) D_2^{k-1} a^{D_1}_{n/D_2}(f) \end{aligned}$$

hence

$$(f, \Omega)_\tau = \sum_{n=1}^{\infty} n^{k-1}(f, G_n)\omega_n^0(z_1, z_2) = \frac{(k-2)!}{(4\pi)^{k-1}} \sum_{D = D_1 D_2} \psi(D_2) D_2^{k-1} \sum_{m=1}^{\infty} a_m^{D_1}(f)\omega^0_{mD_2}(z_1, z_2) \tag{4.61}$$

Using the Fourier expansion

$$\omega_m^0(z_1, z_2) = \frac{2(2\pi)^k}{(k-1)!}(-1)^{\frac{k}{2}} \sum_{\substack{\lambda \in \mathfrak{d}^{-1} \\ \lambda \gg 0, N(\lambda) = m/D}} \sum_{r=1}^{\infty} r^{k-1} e^{2\pi i (r\lambda z_1 + r\lambda' z_2)}$$

leads to the following result:

Theorem 4.5.1. *Let f be a cusp form of weight k for $\Gamma_0(D)$ with character ε, and let $a_n^{D_1}(f)$ be its Fourier coefficients. For each integral ideal $\mathfrak{a}$ of $\mathcal{O}_K$, define*

$$c(\mathfrak{a}) = \sum_{r | \mathfrak{a}} r^{k-1} \sum_{D_2 | (D, N(\mathfrak{a})/r^2)} \psi(D_2) D_2^{k-1} a^{D_1}_{N(\mathfrak{a})/r^2 D_2}(f) \tag{4.62}$$

Then the series

$$F(z_1, z_2) = \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} c((\nu)\mathfrak{d}) e^{2\pi i (\nu z_1 + \nu' z_2)} \tag{4.63}$$

is a cusp form of weight k for $\mathrm{SL}_2(\mathcal{O}_K)$. Moreover, the map $f \mapsto F$ is, up to a constant factor, the map sending f to its inner product with $\Omega(z_1, z_2; \tau)$.

This result is closely related to the Doi-Naganuma lift, which we now describe: let

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$$

be a cusp form of weight k for $\Gamma_0(D)$, where now $D = p$ is a prime such that $K = \mathbb{Q}(\sqrt{D})$ has class number 1. Moreover, assume that f is an eigenfunction for the Hecke operators T_n, normalized so that $a_1 = 1$. If

$$L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}$$

is the L-function of f (with the sum converging in a suitable half-plane), then $L(s, f)$ has an Euler product

$$L(s, f) = \prod_q \left(1 - a_q q^{-s} + \left(\frac{q}{p}\right) q^{k-1-2s}\right)^{-1} \tag{4.64}$$

with the product taken over all primes q. If we define $a_n^\rho = \overline{a_n}$, then

$$f^\rho(z) = \sum_{n=1}^{\infty} a_n^\rho e^{2\pi i n z} \tag{4.65}$$

is also a modular form, with corresponding L-function $L(s, f^\rho)$.

Hecke proved that if $(\frac{q}{p}) = 1$ then a_q is real, if $(\frac{q}{p}) = -1$ then a_q is purely imaginary, and that if $q = p$ then $|a_q| = p^{\frac{k-1}{2}}$. This leads to the following:

Lemma 4.5.1.

$$\Phi(s) = L(s, f)L(s, f^\rho) = \prod_{\mathfrak{q}} \left(1 - b(\mathfrak{q}) N(\mathfrak{q})^{-s} + N(\mathfrak{q})^{k-1-2s}\right)^{-1} \tag{4.66}$$

where now the product is taken over prime ideals $\mathfrak{q}$ of $K = \mathbb{Q}(\sqrt{p})$, and where

$$b(\mathfrak{q}) = a_q \qquad \mathfrak{q}\mathfrak{q}' = (q),\ \left(\frac{q}{p}\right) = 1$$

$$b(\mathfrak{q}) = a_q^2 + 2q^{k-1} \qquad \mathfrak{q} = (q),\ \left(\frac{q}{p}\right) = -1$$

$$b(\mathfrak{q}) = a_p + \overline{a_p} \qquad \mathfrak{q}^2 = (p)$$

Indeed, if $\mathfrak{q}\mathfrak{q}' = (q)$ and $(\frac{q}{p}) = 1$ then we know that $a_q = \overline{a_q}$, so the factor $(1 - a_q q^{-s} + q^{k-1-2s})^{-1}$ appears twice in $L(s,f)L(s,f^\rho)$. And since there are two primes with norm q, it appears twice in the product on the right as well.

Similarly, if $\mathfrak{q} = (q)$ and $(\frac{q}{p}) = -1$ then $\overline{a_q} = -a_q$, so the corresponding factor in Φ is

$$(1 - a_q q^{-s} + q^{k-1-2s})^{-1}(1 + a_q q^{-s} + q^{k-1-2s})^{-1}$$

$$= (1 - a_q^2 q^{-2s} - 2q^{k-1-2s} + q^{2k-2-4s})^{-1}$$

$$= (1 - b(\mathfrak{q})N(\mathfrak{q})^{-s} + N(\mathfrak{q})^{k-1-2s})^{-1}$$

with $b(\mathfrak{q})$ defined as above.

Finally, if $\mathfrak{q}^2 = (p)$, then the factor in Φ is

$$(1 - a_p p^{-s})^{-1}(1 - \overline{a_p} p^{-s})^{-1} = (1 - (a_p + \overline{a_p})p^{-s} + p^{k-1-2s})^{-1}$$

$$= (1 - b(\mathfrak{q})N(\mathfrak{q})^{-s} + N(\mathfrak{q})^{k-1-2s})^{-1}$$

We now want to define $b(\mathfrak{a})$ for all integral ideals $\mathfrak{a}$. Begin with prime powers by writing

$$(1 - b(\mathfrak{q})t + N(\mathfrak{q})^{k-1}t^2)^{-1} = 1 + \sum_{r=1}^{\infty} b(\mathfrak{q}^r)t^r \tag{4.67}$$

as formal power series in $\mathbb{R}[[t]]$. Now that we have defined $b(\mathfrak{q}^r)$ for all primes $\mathfrak{q}$, we extend b to integral ideals by requiring it to be multiplicative. With this definition, we have

$$\Phi(s) = \sum_{\mathfrak{a}} b(\mathfrak{a})N(\mathfrak{a})^{-s} \tag{4.68}$$

We can now state the result of Doi and Naganuma:

Theorem 4.5.2. *Let $p \equiv 1 \pmod 4$ be a prime such that $K = \mathbb{Q}(\sqrt{p})$ has class number 1, and f a normalized Hecke eigenform of weight k for $\Gamma_0(p)$ and ε. If b is defined as above, then*

$$F(z_1, z_2) = \sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} b((\nu)\mathfrak{d})e^{2\pi i(\nu z_1 + \nu' z_2)} \tag{4.69}$$

satisfies

$$F\Big(-\frac{1}{z_1}, -\frac{1}{z_2}\Big) = (z_1 z_2)^k F(z_1, z_2)$$

Since F is also periodic and is invariant under $(z_1, z_2) \mapsto (\lambda z_1, \lambda' z_2)$ for λ a totally positive unit, it follows from a result of Vaserstein that F is a Hilbert modular form.

We will devote the rest of this section to using the function $\Omega(z_1, z_2; \tau)$ to give an alternate proof of this theorem of Doi and Naganuma.

Specifically, we will prove the following:

Theorem 4.5.3. *With p as in Theorem 4.5.2, the function F defined in (4.69) is precisely the function*

$$\sum_{\substack{\nu \in \mathfrak{d}^{-1} \\ \nu \gg 0}} c((\nu)\mathfrak{d}) e^{2\pi i(\nu z_1 + \nu' z_2)}$$

defined in (4.63) (and so we are justified in calling both F).

To prove this claim, it is enough to show that $b(\mathfrak{a}) = c(\mathfrak{a})$ for all integral ideals $\mathfrak{a}$. Since $D = p$ is prime, either $D_2 = 1$ or $D_2 = p$, hence now

$$c(\mathfrak{a}) = \sum_{r|\mathfrak{a}} r^{k-1} a^p_{N(\mathfrak{a})/r^2}(f) + \sum_{r\mathfrak{d}|\mathfrak{a}} r^{k-1} p^{k-\frac{1}{2}} a^1_{N(\mathfrak{a})/r^2 p}(f) \tag{4.70}$$

using $\psi(1) = 1$ and $\psi(p) = \sqrt{p}$.

The Fourier coefficients $a^p_n(f)$ are the Fourier coefficients of $f|_k \begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix}$, and since this matrix is in $\Gamma_0(p)$, these are just the a_n in the definition of f.

Similarly, the Fourier coefficients $a^1_n(f)$ are the coefficients of $f|_k \begin{bmatrix} p & -1 \\ 1 & 0 \end{bmatrix}$, so are given by

$$(z-p)^{-k} f\Big(-\frac{1}{z-p}\Big) = \sum_{n=1}^{\infty} a^1_n(f) e^{\frac{2\pi i n z}{p}} \tag{4.71}$$

Replacing $z - p$ with pz, this becomes

$$p^{-k} z^{-k} f\Big(-\frac{1}{pz}\Big) = \sum_{n=1}^{\infty} a^1_n(f) e^{2\pi i n z} \tag{4.72}$$

On the other hand, lemma 2 in Naganuma [Nag73] proves that

$$p^{-k} z^{-k} f\Big(-\frac{1}{pz}\Big) = p^{\frac{1}{2}-k} \overline{a_p} f^{\rho}(z) \tag{4.73}$$

and so

$$a_n^1(f) = p^{\frac{1}{2}-k}\overline{a_p a_n} \tag{4.74}$$

And from examining $L(s, f^\rho)$ we see that $\overline{a_n a_p} = \overline{a_{np}}$. Hence

$$c(\mathfrak{a}) = \sum_{r|\mathfrak{a}} r^{k-1} a'_{N(\mathfrak{a})/r^2} \tag{4.75}$$

with $a'_n = a_n$ if $(p, n) = 1$ and $a'_n = a_n + \overline{a_n}$ if $p|n$.

Since $c(\mathfrak{a}_1\mathfrak{a}_2) = c(\mathfrak{a}_1)c(\mathfrak{a}_2)$ if the norms of $\mathfrak{a}_1$ and $\mathfrak{a}_2$ are coprime, it is enough to prove that $c(\mathfrak{a}) = b(\mathfrak{a})$ for integral ideals $\mathfrak{q}^m$ with $\mathfrak{q}$ either inert or ramified, or for products $\mathfrak{q}^r(\mathfrak{q}')^r$ with $\mathfrak{q}\mathfrak{q}' = (q)$.

First consider the case $\mathfrak{a} = \mathfrak{q}^m$ with $\mathfrak{q} = (q)$ and $(\frac{q}{p}) = -1$. In this case $r|\mathfrak{a}$ when $r = 1, q, \ldots, q^m$, so

$$c(\mathfrak{q}^m) = \sum_{j=0}^{m} q^{j(k-1)} a_{q^{2m-2}}$$

To evaluate this sum, we introduce a formal power series and obtain

$$\begin{aligned}
\sum_{m=0}^{\infty} c(\mathfrak{q}^m)t^{2m} &= \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} q^{j(k-1)} a_{q^{2l}} t^{2l+2j} \\
&= \Big(\sum_{j=0}^{\infty} q^{j(k-1)} t^{2j}\Big)\Big(\sum_{l=0}^{\infty} a_{q^{2l}} t^{2l}\Big) \\
&= (1 - q^{k-1}t^2)^{-1}\Big(\frac{1}{2}\sum_{l=0}^{\infty} a_{q^l} t^l + \frac{1}{2}\sum_{l=0}^{\infty} a_{q^l}(-t)^l\Big) \\
&= \frac{1}{2}(1 - q^{k-1}t^2)^{-1}\Big(\frac{1}{1 - a_q t - q^{k-1}t^2} + \frac{1}{1 + a_q t - q^{k-1}t^2}\Big) \\
&= \frac{1}{(1 - a_q t - q^{k-1}t^2)(1 + a_q t - q^{k-1}t^2)} = \frac{1}{1 - b(\mathfrak{q})t^2 + q^{2k-2}t^4} \\
&= \sum_{m=0}^{\infty} b(\mathfrak{q}^m)t^{2m}
\end{aligned}$$

This handles the first case.

For the second case, suppose that $\mathfrak{a} = \mathfrak{q}^m$ with $\mathfrak{q}^2 = (p)$. Now r divides $\mathfrak{a}$ when $r = 1, p, \ldots, p^{\lfloor \frac{m}{2} \rfloor}$, so

$$c(\mathfrak{q}^m) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} p^{j(k-1)} a'_{p^{m-2}} = \sum_{0 \le j \le \frac{m}{2}} p^{j(k-1)} a_{p^{m-2}} + \sum_{0 \le j \le \frac{m-1}{2}} p^{j(k-1)} \overline{a_{p^{m-1}}}$$

Again we use a formal power series, and using the fact that $a_{p^m} = (a_p)^m$, we get

$$\sum_{m=0}^{\infty} c(\mathfrak{q}^m)t^m = \sum_{j=0}^{\infty}\sum_{l=0}^{\infty} p^{j(k-1)}a_{p^l}t^{l+2j} + \sum_{j=0}^{\infty}\sum_{l=1}^{\infty} p^{j(k-1)}\overline{a_{p^l}}t^{l+2j}$$

$$= \Big(\sum_{j=0}^{\infty} p^{j(k-1)}t^{2j}\Big)\Big(\sum_{l=0}^{\infty} a_p^l t^l\Big) + \Big(\sum_{j=0}^{\infty} p^{j(k-1)}t^{2j}\Big)\Big(\sum_{l=1}^{\infty} \overline{a_p}^l t^l\Big)$$

$$= \frac{1}{1-p^{k-1}t^2}\Big(\frac{1}{1-a_pt} + \frac{\overline{a_p}}{1-\overline{a_p}t}\Big)$$

And since $a_p\overline{a_p} = p^{k-1}$, this becomes

$$\frac{1}{1-p^{k-1}t^2}\frac{1-a_p\overline{a_p}t^2}{(1-a_pt)(1-\overline{a_p}t)} = \frac{1}{1-(a_p+\overline{a_p})t + p^{k-1}t^2}$$

$$= \frac{1}{1-b(\mathfrak{q})t+p^{k-1}t^2} = \sum_{m=0}^{\infty} b(\mathfrak{q}^m)t^m$$

so in this case we also have $c(\mathfrak{q}^m) = b(\mathfrak{q}^m)$.

Finally, suppose that $\mathfrak{a} = \mathfrak{q}^m(\mathfrak{q}')^n$ with $\mathfrak{q}\mathfrak{q}' = (q)$ and $(\frac{q}{p}) = 1$. In this case r divides $\mathfrak{a}$ for $r = 1, q, \ldots, q^{\min\{m,n\}}$, so

$$c(\mathfrak{q}^m(\mathfrak{q}')^n) = \sum_{j=0}^{\min\{m,n\}} q^{j(k-1)}a_{q^{m+n-2j}}$$

In this case we use a formal power series in two variables:

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c(\mathfrak{q}^m(\mathfrak{q}')^n)t^m u^n = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} q^{i(k-1)}a_{q^{j+l}}t^{i+j}u^{i+l}$$

$$= \Big(\sum_{i=0}^{\infty} q^{i(k-1)}t^iu^i\Big)\Big(\sum_{n=0}^{\infty} a_{q^n}(t^n + t^{n-1}u + \cdots + tu^{n-1} + u^n)\Big)$$

$$= \frac{1}{1-q^{k-1}tu}\frac{1}{t-u}\sum_{n=0}^{\infty} a_{q^n}(t^{n+1}-u^{n+1}) = \frac{1}{1-q^{k-1}tu}\frac{1}{t-u}\Big(t\sum_{n=0}^{\infty} a_{q^n}t^n - u\sum_{n=0}^{\infty} a_{q^n}u^n\Big)$$

$$= \frac{1}{1-q^{k-1}tu}\frac{1}{t-u}\Big(\frac{t}{1-a_qt+q^{k-1}t^2} - \frac{u}{1-a_qu+q^{k-1}u^2}\Big)$$

$$= \frac{1}{(1-a_qt+q^{k-1}t^2)(1-a_qu+q^{k-1}u^2)} = \frac{1}{1-b(\mathfrak{q})t+N(\mathfrak{q})^{k-1}t^2}\frac{1}{1-b(\mathfrak{q}'u+N(\mathfrak{q}')^{k-1}u^2}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b(\mathfrak{q}^m)b((\mathfrak{q}')^n)t^mu^n = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b(\mathfrak{q}^m(\mathfrak{q}')^n)t^mu^n$$

with the last equality following from the fact that $\mathfrak{q}$ and $\mathfrak{q}'$ are coprime and b is multiplicative. This completes the proof that $\Omega(z_1, z_2; \tau)$ is a kernel function for the Doi-Naganuma lift.

4.6 Restriction to the diagonal

A Hilbert modular form of weight k for a quadratic field K defines an elliptic modular form of weight $2k$ by restriction to the diagonal $z_1 = z_2$. In this section we will investigate the modular forms obtained when the forms ω_m are restricted to the diagonal, modifying Zagier's original exposition to more closely resemble the cubic case to be covered later.

Once again let $k \geq 4$ be an even integer and $m \geq 0$ an integer, and now write

$$\omega_m(z_1, z_2) = \sideset{}{'}\sum_{\substack{a,c\in\mathbb{Z},\lambda\in\mathfrak{d}^{-1} \\ \lambda\lambda'-ac=m/D}} (az_1z_2 + \lambda z_1 + \lambda' z_2 + c)^{-k}$$

where as usual the sum omits $\lambda = a = c = 0$ if $m = 0$. Setting $z_1 = z_2 = z$, we get

$$\omega_m(z, z) = \sideset{}{'}\sum_{\substack{a,c\in\mathbb{Z},\lambda\in\mathfrak{d}^{-1} \\ \lambda\lambda'-ac=m/D}} (az^2 + \mathrm{tr}(\lambda)z + c)^{-k} \tag{4.76}$$

Set $b = \mathrm{tr}(\lambda)$, and note that b is an integer since $\lambda \in \mathfrak{d}^{-1}$. Also define $d = \sqrt{D}(\lambda - \lambda')$, and observe that d is also an integer since $\mathfrak{d}^{-1} = \frac{1}{\sqrt{D}}\mathcal{O}_K$. Then

$$\lambda = \frac{1}{2}(\lambda + \lambda') + \frac{1}{2}(\lambda - \lambda') = \frac{b}{2} + \frac{d}{2\sqrt{D}} \tag{4.77}$$

and

$$ac - \lambda\lambda' = ac - \frac{b^2}{4} + \frac{d^2}{D} \tag{4.78}$$

hence

$$b^2 - 4ac = \frac{d^2 + 4m}{D} \tag{4.79}$$

Therefore we have

$$\omega_m(z, z) = \sideset{}{'}\sum_{\substack{a,b,c,d\in\mathbb{Z} \\ d^2-D(b^2-4ac)=-4m}} (az^2 + bz + c)^{-k} = \sum_{\substack{d\in\mathbb{Z} \\ d^2\equiv-4m\ (D)}} f_k\Big(\frac{d^2 + 4m}{D}, z\Big)$$

where

$$f_k(\Delta, z) = \sideset{}{'}\sum_{\substack{a,b,c\in\mathbb{Z} \\ b^2-4ac=\Delta}} (az^2 + bz + c)^{-k} \tag{4.80}$$

for $\Delta \geq 0$.

The functions $f_k(\Delta, z)$ are an analogue of the forms ω_m for $\mathrm{SL}_2(\mathbb{Z})$. Let us first show that they transform like modular forms of weight $2k$. If

$$\gamma = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

then

$$f_k(\Delta, \gamma z) = (rz+s)^{2k} \sideset{}{'}\sum_{\substack{a^*, b^*, z^* \in \mathbb{Z} \\ (b^*)^2 - 4a^*c^* = \Delta}} (a^* z^2 + b^* z + c^*)^{-k}$$

where the integers a^*, b^*, c^* are defined by

$$\begin{bmatrix} a^* & \frac{b^*}{2} \\ \frac{b^*}{2} & c^* \end{bmatrix} = \gamma^T \begin{bmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{bmatrix} \gamma$$

From this we see that in fact $(b^*)^2 - 4a^*c^* = \Delta$ since γ has determinant 1, and moreover (a^*, b^*, c^*) runs over the same set as (a, b, c). Therefore $f(\Delta, z)$ transforms like a modular form of weight $2k$.

To prove that $f_k(\Delta, z)$ is in fact a modular form, we must show that it is holomorphic at infinity. The proof breaks into the cases $\Delta = 0$ and $\Delta > 0$.

For $\Delta = 0$, the identity $b^2 = 4ac$ is homogeneous in a, b, and c, so factoring out the greatest common divisor yields

$$f_k(0, z) = \zeta(k) \sum_{\substack{a,b,c \in \mathbb{Z} \\ \gcd(a,b,c)=1 \\ b^2 = 4ac}} (az^2 + bz + c)^{-k} \tag{4.81}$$

Next, if $b^2 = 4ac$ and $\gcd(a, b, c) = 1$, then $\gcd(a, c) = 1$, and there are relatively prime integers m and n such that $a = \pm m^2, c = \pm n^2, b = 2mn$ (with the signs of a and c the same). Hence

$$f_k(0, z) = \zeta(k) \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n)=1}} (mz+n)^{-2k} = \frac{\zeta(k)}{\zeta(2k)} G_{2k}(z) \tag{4.82}$$

where

$$G_{2k}(z) = \sum_{m,n \in \mathbb{Z}} (mz+n)^{-2k}$$

is the usual Eisenstein series of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z})$. This shows that $f_k(0,z)$ is a modular form.

For $\Delta > 0$, we will show that $f_k(\Delta, z)$ is a cusp form by explicitly computing the Fourier coefficients. Begin by writing

$$f_k(\Delta, z) = f_k^0(\Delta, z) + 2\sum_{a=1}^{\infty} f_k^a(\Delta, z) \tag{4.83}$$

where

$$f_k^a(\Delta, z) = \sum_{\substack{b,c\in\mathbb{Z} \\ b^2-4ac=\Delta}} (az^2+bz+c)^{-k} \tag{4.84}$$

for fixed $a \geq 0$. For $a = 0$, we have

$$f_k^0(\Delta, z) = \sum_{\substack{b\in\mathbb{Z} \\ b^2=\Delta}} \sum_{c\in\mathbb{Z}} (bz+c)^{-k} = \sum_{\substack{b\in\mathbb{Z} \\ b^2=\Delta}} \frac{(2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1} e^{2\pi i r b z} \tag{4.85}$$

(using the "well-known" Fourier expansion in (4.15)).

If $a \geq 1$, then

$$f_k^a(\Delta, z) = \sum_{\substack{b\in\mathbb{Z} \\ b^2\equiv\Delta\ (4a)}} \left(az^2+bz+\frac{b^2-\Delta}{4a}\right)^{-k} = \sum_{\substack{b \bmod 2a \\ b^2\equiv\Delta\ (4a)}} \sum_{n\in\mathbb{Z}} \left(a(z+n)^2+b(z+n)+\frac{b^2-\Delta}{4a}\right)^{-k}$$

As the first sum is finite, it's enough to compute the Fourier coefficients of the inner sum for fixed a, b with $a \geq 1$. By the usual unfolding argument, these are given by

$$\int_i^{i+1} \sum_{n\in\mathbb{Z}} \left(a(z+n)^2+b(z+n)+\frac{b^2-\Delta}{4a}\right)^{-k} e^{-2\pi i r z}\, dz = \int_{i-\infty}^{i+\infty} \left(az^2+bz+\frac{b^2-\Delta}{4a}\right)^{-k} e^{-2\pi i r z}\, dz \tag{4.86}$$

The poles of the integrand lie on the real axis, so if $r \leq 0$ then we can shift contours to $i\infty$ and conclude that the rth Fourier coefficient is zero. This proves that $f_k(\Delta, z)$ is a cusp form if $\Delta > 0$.

Next, suppose that $r \geq 1$. While $f_k(\Delta, z)$ is defined for even integers $k \geq 4$, the integral in (4.86) converges for $k \geq 1$, and we will begin with $k = 1$. Write

$$az^2+bz+\frac{b^2-\Delta}{4a} = a(z-r_1)(z-r_2) \tag{4.87}$$

with

$$r_1 = \frac{-b+\sqrt{\Delta}}{2a} \qquad r_2 = \frac{-b-\sqrt{\Delta}}{2a} \tag{4.88}$$

Shifting contours downward to $-i\infty$, it follows from the residue theorem that

$$\int_{i-\infty}^{i+\infty} \Big(az^2 + bz + \frac{b^2-\Delta}{4a}\Big)^{-1} e^{-2\pi irz}\, dz$$

$$= -\frac{2\pi i}{a}\Big(\mathrm{res}_{z=r_1}\frac{e^{-2\pi irz}}{(z-r_1)(z-r_2)} + \mathrm{res}_{z=r_2}\frac{e^{-2\pi irz}}{(z-r_1)(z-r_2)}\Big)$$

(with the negative because the contour is traversed clockwise)

$$= -\frac{2\pi i}{a}\Big(\frac{e^{-2\pi irr_1}}{r_1-r_2} + \frac{e^{-2\pi irr_2}}{r_2-r_1}\Big) = -\frac{2\pi i}{a} e^{-\pi ir(r_1+r_2)}\frac{e^{-\pi ir(r_1-r_2)} - e^{\pi ir(r_1-r_2)}}{r_1-r_2}$$

$$= -\frac{4\pi^2}{a} e^{-\pi ir(r_1+r_2)}\frac{\sin(\pi r(r_1-r_2))}{\pi(r_1-r_2)}$$

Using $r_1 + r_2 = -\frac{b}{a}$ and $r_1 - r_2 = \frac{\sqrt{\Delta}}{a}$, we conclude that

$$\int_{i-\infty}^{i+\infty} \Big(az^2 + bz + \frac{b^2-\Delta}{4a}\Big)^{-1} e^{-2\pi irz}\, dz = -\frac{4\pi^2 r}{a} e^{\frac{\pi irb}{a}} \frac{\sin(\frac{\pi r\sqrt{\Delta}}{a})}{\frac{\pi r\sqrt{\Delta}}{a}}$$

$$= -\frac{4\pi^2 r}{a} e^{\frac{\pi irb}{a}} \frac{\sin t}{t}$$

for $t = \frac{\pi r\sqrt{\Delta}}{a}$.

Finally, to evaluate the integral

$$\int_{i-\infty}^{i+\infty} \Big(az^2 + bz + \frac{b^2-\Delta}{4a}\Big)^{-k} e^{-2\pi irz}\, dz \tag{4.89}$$

for all integers $k \geq 1$, we view t as a positive real parameter, and differentiate $k-1$ times to obtain

$$\int_{i-\infty}^{i+\infty} \Big(az^2 + bz + \frac{b^2-\Delta}{4a}\Big)^{-k} e^{-2\pi irz}\, dz = \int_{i-\infty}^{i+\infty} \Big(az^2 + bz + \frac{b^2}{4a} - \frac{at^2}{(2\pi r)^2}\Big)^{-k} e^{-2\pi irz}\, dz$$

$$= \frac{(2\pi r)^{2k-2}}{a^{k-1}(k-1)!}\frac{d^{k-1}}{d(t^2)^{k-1}} \int_{i-\infty}^{i+\infty} \Big(az^2 + bz + \frac{b^2}{4a} - \frac{at^2}{(2\pi r)^2}\Big)^{-1} e^{-2\pi irz}\, dz$$

$$= -\frac{(2\pi r)^{2k-2}}{a^{k-1}(k-1)!}\frac{4\pi^2 r}{a} e^{\frac{\pi irb}{a}} \frac{d^{k-1}}{d(t^2)^{k-1}}\Big(\frac{\sin t}{t}\Big)$$

and using the identity

$$J_{k-\frac{1}{2}}(t) = \frac{(-1)^{k-1}(2t)^{k-\frac{1}{2}}}{\sqrt{\pi}} \frac{d^{k-1}}{d(t^2)^{k-1}}\Big(\frac{\sin t}{t}\Big) \tag{4.90}$$

we finally obtain

$$\begin{aligned}
&\frac{(2\pi r)^{2k-2}}{a^{k-1}(k-1)!}\frac{4\pi^2 r}{a}e^{\frac{\pi irb}{a}}\frac{\sqrt{\pi}}{(\frac{2\pi r\sqrt{\Delta}}{a})^{k-\frac{1}{2}}}J_{k-\frac{1}{2}}\Big(\frac{\pi r\sqrt{\Delta}}{a}\Big)\\
&= \frac{2^{k+\frac{1}{2}}\pi^{k+1}r^{k-\frac{1}{2}}}{\Delta^{\frac{k}{2}-\frac{1}{4}}\sqrt{a}(k-1)!}e^{\frac{\pi irb}{a}}J_{k-\frac{1}{2}}\Big(\frac{\pi r\sqrt{\Delta}}{a}\Big)
\end{aligned}$$

Summing over a, we now have

$$f_k(\Delta, z) = \sum_{r=1}^{\infty} c(r,\Delta)e^{2\pi irz} \tag{4.91}$$

with

$$c(r,\Delta) = \frac{(2\pi i)^k}{(k-1)!}\sum_{r^2=d^2\Delta} d^{k-1} + \frac{2^{k+\frac{3}{2}}\pi^{k+1}r^{k-\frac{1}{2}}}{\Delta^{\frac{k}{2}-\frac{1}{4}}(k-1)!}\sum_{a=1}^{\infty} a^{-\frac{1}{2}}J_{k-\frac{1}{2}}\Big(\frac{\pi r\sqrt{\Delta}}{a}\Big)\sum_{\substack{b \mod 2a\\ b^2\equiv\Delta\ (4a)}} e^{\frac{\pi irb}{a}} \tag{4.92}$$

CHAPTER 5

Modular Forms for Cubic Fields

In this chapter we construct a family of Hilbert modular forms for totally real cubic fields which is analogous to the modular forms defined by Don Zagier and covered in chapter four. Before getting to the definition, we review some results involving hypermatrices and hyperdeterminants.

5.1 Tensors and hypermatrices

If V_1, V_2, V_3 are real vector spaces, their tensor product $V_1 \otimes V_2 \otimes V_3$ is a real vector space consisting of all linear combinations of "pure" tensors $v_1 \otimes v_2 \otimes v_3$ with $v_j \in V_j$. (Of course one can define a tensor product of some other number of vector spaces, but we will be interested in the case of three vector spaces.)

The tensor product $V_1 \otimes V_2 \otimes V_3$ is characterized by the following universal property: if $\Phi : V_1 \times V_2 \times V_3 \to V_1 \otimes V_2 \otimes V_3$ is the map

$$\Phi(v_1, v_2, v_3) = v_1 \otimes v_2 \otimes v_3$$

then for every trilinear map $T : V_1 \times V_2 \times V_3 \to W$ there is a unique linear map $t : V_1 \otimes V_2 \otimes V_3 \to W$ such that $t \circ \Phi = T$.

If V_j has dimension d_j for $j = 1, 2, 3$, then the tensor product $V_1 \otimes V_2 \otimes V_3$ has dimension $d_1 d_2 d_3$. If we choose bases $\{e_i\}_{i=1}^{d_1}, \{f_j\}_{j=1}^{d_2}, \{g_k\}_{k=1}^{d_3}$ for V_1, V_2, V_3, then the set

$$\{e_i \otimes f_j \otimes g_k : 1 \le i \le d_1, 1 \le j \le d_2, 1 \le k \le d_3\}$$

is a basis for $V_1 \otimes V_2 \otimes V_3$.

This means that for every element of $V_1 \otimes V_2 \otimes V_3$ there is a corresponding $d_1 \times d_2 \times d_3$ hypermatrix $A \in \mathbb{R}^{d_1 \times d_2 \times d_3}$. Such a hypermatrix can be represented by an array $A = [a_{ijk}]$ where $1 \leq i \leq d_1, 1 \leq j \leq d_2$, and $1 \leq k \leq d_3$. Addition and scalar multiplication of hypermatrices are defined coordinatewise.

Matrix multiplication is a bit more interesting. Recall that if A is a $d_1 \times d_2$ matrix then it can be multiplied on the left by a $c_1 \times d_1$ matrix or on the right by a $d_2 \times e_2$ matrix. A hypermatrix can be multiplied by a matrix on 3 "sides":

If $L = [\lambda_{pi}]$ is a $c_1 \times d_1$ matrix, then the product of L and A is the hypermatrix $L \cdot A = A' = [a'_{pjk}]$ with

$$a'_{pjk} = \sum_{i=1}^{d_1} \lambda_{pi} a_{ijk} \tag{5.1}$$

Similarly, if $M = [\mu_{qj}]$ is a $c_2 \times d_2$ matrix, then $M \cdot A = A' = [a'_{iqk}]$ with

$$a'_{iqk} = \sum_{j=1}^{d_2} \mu_{qj} a_{ijk} \tag{5.2}$$

and if $N = [\nu_{rk}]$ is a $c_3 \times d_3$ matrix, then $N \cdot A = A' = [a'_{ijr}]$ with

$$a'_{ijr} = \sum_{k=1}^{d_3} \nu_{rk} a_{ijk} \tag{5.3}$$

These three formulas can be neatly combined by defining

$$(L, M, N) \cdot A = A' = [a'_{pqr}] \tag{5.4}$$

with

$$a'_{pqr} = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} \sum_{k=1}^{d_3} \lambda_{pi} \mu_{qj} \nu_{rk} a_{ijk} \tag{5.5}$$

This multiplication is linear in the sense that

$$(L, M, N) \cdot (\alpha_1 A_1 + \alpha_2 A_2) = \alpha_1 (L, M, N) \cdot A_1 + \alpha_2 (L, M, N) \cdot A_2$$

for real numbers α_1, α_2 and hypermatrices A_1, A_2, and

$$(\alpha_1 L_1 + \alpha_2 L_2, M, N) \cdot A = \alpha_1 (L_1, M, N) \cdot A + \alpha_2 (L_2, M, N) \cdot A$$

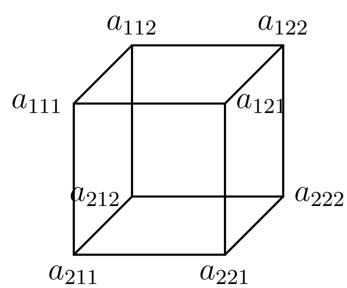

Figure 5.1: A hypermatrix represented as a cube

and similarly for M and N. The multiplication is associative in that

$$(L', M', N') \cdot [(L, M, N) \cdot A] = (L'L, M'M, N'N) \cdot A$$

provided that the matrices have the appropriate dimensions.

5.2 $2 \times 2 \times 2$ hypermatrices and the hyperdeterminant

In the case $d_1 = d_2 = d_3 = 2$, a hypermatrix $A = [a_{ijk}]$ can be visualized as a cube as in Figure 5.1.

By "slicing" the cube along a horizontal plane, we can also view A as follows:

$$A = \left[\begin{array}{cc|cc} a_{111} & a_{112} & a_{211} & a_{212} \\ a_{121} & a_{122} & a_{221} & a_{222} \end{array}\right] = [A_1 \mid A_2] \tag{5.6}$$

We can then define a binary quadratic form

$$\det(A_1X + A_2Y) = \det(A_1)X^2 + \frac{\det(A_1 + A_2) - \det(A_1 - A_2)}{2}XY + \det(A_2)Y^2 \tag{5.7}$$

The hyperdeterminant is then defined to be the discriminant of this polynomial:

$$\operatorname{hypdet}(A) = \Big(\frac{\det(A_1 + A_2) - \det(A_1 - A_2)}{2}\Big)^2 - 4\det(A_1)\det(A_2) \tag{5.8}$$

$$= a_{111}^2a_{222}^2 + a_{112}^2a_{221}^2 + a_{121}^2a_{212}^2 + a_{122}^2a_{211}^2 - 2(a_{111}a_{112}a_{221}a_{222} + a_{111}a_{121}a_{212}a_{222}$$

$$+a_{111}a_{122}a_{211}a_{222} + a_{112}a_{121}a_{212}a_{221} + a_{112}a_{122}a_{221}a_{211} + a_{121}a_{122}a_{212}a_{211})$$

$$+4(a_{111}a_{122}a_{212}a_{221} + a_{112}a_{121}a_{211}a_{222})$$

The cube can of course be cut along two other planes; doing so leads to different quadratic forms, but all three quadratic forms have the same discriminant.

This fact has a useful consequence: the hyperdeterminant is invariant under the two "transpositions" of the hypermatrix A given by the cyclic permutations $a_{ijk} \mapsto a_{kij}$ and $a_{ijk} \mapsto a_{jki}$.

Moreover, if L, M, and N are 2×2 matrices, then

$$\operatorname{hypdet}((L, M, N) \cdot A) = \det(L)^2 \det(M)^2 \det(N)^2 \operatorname{hypdet}(A) \tag{5.9}$$

This can be proved as follows: using the representation above we have

$$(I, M, I) \cdot [A_1 \mid A_2] = [MA_1 \mid MA_2] \tag{5.10}$$

hence

$$\operatorname{hypdet}((I, M, I) \cdot A) = \det(M)^2 \operatorname{hypdet}(A)$$

By slicing the cube along the other two planes we obtain the analogous result for $(L, I, I) \cdot A$ and $(I, I, N) \cdot A$, so combining these identities proves the general result.

In particular, the hyperdeterminant is invariant under the action of $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$.

Finally, referring back to Figure 5.1, we note that the terms in equation (5.8) have the following interpretation: the four terms in the first group correspond to the four main diagonals of the cube, the six terms in the second group correspond to the six rectangles formed by pairs of opposite edges, and the two terms in the third group correspond to the two tetrahedra whose edges are the diagonals of the faces of the cube.

5.3 A geometric interpretation of the hyperdeterminant

In this section we briefly describe an interesting geometric interpretation of the $2 \times 2 \times 2$ hyperdeterminant. More details and proofs can be found in [GKZ08].

Let V be a finite-dimensional complex vector space, and denote by $\mathbb{P}(V)$ the corresponding projective space. The set of all hyperplanes in $\mathbb{P}(V)$ forms another projective space $\mathbb{P}(V)^*$, which can be identified with $\mathbb{P}(V^*)$, where V^* is the dual space of V.

If $X \subset \mathbb{P}(V)$ is a closed irreducible algebraic subvariety, define $X^\vee \subset \mathbb{P}(V)^*$ to be the closure of the set of all hyperplanes tangent to X. If X is irreducible then $X^\vee$ is as well.

Now suppose that $V = \mathbb{C}^{2\times 2}$, and identify V with its dual via the bilinear form

$$(A, B) = \sum_{i,j} A_{ij} B_{ij}$$

If $X_r \subset \mathbb{P}(V)$ is the projectivization of the subspace of V consisting of matrices of rank at most r, then one can show that $X_r^\vee = X_{2-r}$. In particular, if $r = 1$ then both X_1 and its dual are defined by the vanishing of the determinant.

We now recall the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$, which is defined by the map

$$([x_1 : x_2], [y_1 : y_2]) \mapsto [x_1y_1 : x_1y_2 : x_2y_1 : x_2y_2]$$

in homogeneous coordinates. One can check that the image of the Segre embedding is precisely X_1, and so its dual is also X_1.

What happens if we instead consider the threefold Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^7$? In this case the dual variety to the image of the Segre embedding is defined by the vanishing of the $2 \times 2 \times 2$ hyperdeterminant. In other words, if the hypermatrix $A = [a_{ijk}]$ has hyperdeterminant zero then the system of six equations

$$a_{111}x_1y_1 + a_{121}x_1y_2 + a_{211}x_2y_1 + a_{221}x_2y_2$$

$$a_{112}x_1y_1 + a_{122}x_1y_2 + a_{212}x_2y_1 + a_{222}x_2y_2$$

$$a_{111}x_1z_1 + a_{112}x_1z_2 + a_{211}x_2z_1 + a_{212}z_2z_2$$

$$a_{121}x_1z_1 + a_{122}x_1z_2 + a_{221}x_2z_1 + a_{222}z_2z_2$$

$$a_{111}y_1z_1 + a_{112}y_1z_2 + a_{121}y_2z_1 + a_{122}y_2z_2$$

$$a_{211}y_1z_1 + a_{212}y_1z_2 + a_{221}y_2z_1 + a_{222}y_2z_2$$

has a non-trivial solution. It turns out that the converse is true as well, though the proof is harder.

5.4 Defining the modular forms

Let K be a totally real cubic field with discriminant $\Delta > 0$, and let $\mathfrak{d}$ be the different ideal of K. For each positive integer m, define

$$H_m(z_1, z_2, z_3) = \sum_{\substack{a,b\in\mathbb{Z} \\ \lambda,\mu\in\mathfrak{d}^{-1} \\ h(a,b,\lambda,\mu)=-m/\Delta^2 \\ \lambda\lambda'-a\mu\gg 0}} (az_1z_2z_3 + \lambda' z_1 z_2 + \lambda z_1 z_3 + \lambda'' z_2 z_3 + \mu z_1 + \mu' z_2 + \mu'' z_3 + b)^{-k} \tag{5.11}$$

Here $z_1, z_2, z_3 \in \mathbb{H}$, and λ' and λ'' are the two conjugates of λ. The condition $\lambda\lambda' - a\mu \gg 0$ means that $\lambda\lambda' - a\mu$ is totally positive, i.e. the image of $\lambda\lambda' - a\mu$ under the three real embeddings of K is positive. If K is Galois then $\mathfrak{d}^{-1}$ is the inverse different of K, while if K is not Galois then $\mathfrak{d}^{-1} = \mathfrak{d}^{-1}_{L/M}$ is the relative inverse different, where L is the Galois closure of K and $M = \mathbb{Q}(\sqrt{\Delta})$. For the sake of simplicity we will focus on the Galois case in the next few sections.

In (5.11), $h(a, b, \lambda, \mu)$ is the hyperdeterminant of the hypermatrix

$$A = \left[\begin{array}{cc|cc} a & \lambda' & \lambda'' & \mu' \\ \lambda & \mu & \mu'' & b \end{array}\right] \tag{5.12}$$

hence

$$h(a, b, \lambda, \mu) = (ab - \lambda\mu' - \lambda'\mu'' + \lambda''\mu)^2 - 4(\lambda\lambda' - a\mu)(\mu'\mu'' - b\lambda'') \tag{5.13}$$

$$= a^2b^2 + \mathrm{tr}((\lambda\mu')^2) - 2[\mathrm{tr}(ab\lambda\mu') + \mathrm{tr}(\lambda\lambda'\mu'\mu'')] + 4[aN(\mu) + bN(\lambda)]$$

This last expression makes it clear that $h(a, b, \lambda, \mu)$ is a rational number, but this can also be seen in the following way: view the 8-tuple $(a, \lambda, \lambda', \lambda'', \mu, \mu', \mu'', b)$ as a hypermatrix A which can be represented as a cube as in Figure 5.2.

The two cyclic permutations of the entries of A correspond to $A \mapsto A'$ and $A \mapsto A''$, where A' and A'' are the hypermatrices obtained by conjugating the entries of A once or twice.

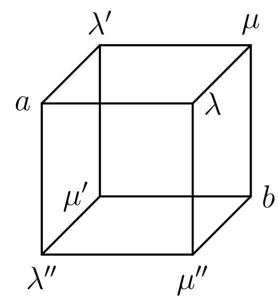

Figure 5.2: The cube corresponding to a, b, λ, μ

Since the hyperdeterminant of A is invariant under these cyclic permutations, it follows that $h(a, b, \lambda, \mu)$ is equal to its two conjugates, hence is a rational number.

The goal of the next few sections is to show that $H_m(z_1, z_2, z_3)$ is a Hilbert modular form of weight k. We will first check that H_m has the appropriate transformation properties under $\mathrm{SL}_2(\mathcal{O}_K)$ under the assumption that the each term in the series is non-zero and that the series converges absolutely. In the following sections we will justify these assumptions, and then in the last section we will show how to adapt the above definition for $m = 0$ and that in this case H_m is a multiple of the Hecke-Eisenstein series of weight k for K.

5.5 The transformation properties of H_m

Suppose that $M = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{SL}_2(\mathcal{O}_K)$, and denote by M' and M'' the matrices obtained by conjugating each entry of M once or twice. If we let these matrices act on $\mathbb{H}$ in the usual way, then

$$H_m(Mz_1, M'z_2, M''z_3)$$
$$= N(\gamma z + \delta)^k \sum_{\substack{a^*, b^* \in \mathbb{Z} \\ \lambda^*, \mu^* \in \mathfrak{d}^{-1}}} (a^* z_1 z_2 z_3 + \lambda^{*\prime} z_1 z_2 + \lambda^* z_1 z_3 + \lambda^{*\prime\prime} z_2 z_3 + \mu^* z_1 + \mu^{*\prime} z_2 + \mu^{*\prime\prime} z_3 + b^*)^{-k}$$

where

$$N(\gamma z + \delta) = (\gamma z_1 + \delta)(\gamma' z_1 + \delta')(\gamma'' z_3 + \delta'')$$

and

$$a^* = aN(\alpha) + \mathrm{tr}(\lambda\alpha\alpha''\gamma') + \mathrm{tr}(\mu\alpha\gamma'\gamma'') + bN(\gamma) \tag{5.14}$$

$$b^* = aN(\beta) + \mathrm{tr}(\lambda\beta\beta''\delta') + \mathrm{tr}(\mu\beta\delta'\delta'') + bN(\delta) \tag{5.15}$$

$$\lambda^* = a\alpha\alpha''\beta' + \lambda'\alpha\beta'\gamma'' + \lambda\alpha\alpha''\delta' + \lambda''\alpha''\beta'\gamma + \mu\alpha\gamma''\delta' + \mu'\beta'\gamma\gamma'' + \mu''\alpha''\gamma\delta' + b\gamma\gamma''\delta' \tag{5.16}$$

$$\mu^* = a\alpha\beta'\beta'' + \lambda'\alpha\beta'\delta'' + \lambda\alpha\beta''\delta' + \lambda''\beta'\beta''\gamma + \mu\alpha\delta'\delta'' + \mu'\beta'\gamma\delta'' + \mu''\beta''\gamma\delta' + b\gamma\delta'\delta'' \tag{5.17}$$

Since $\alpha, \beta, \gamma, \delta$ are algebraic integers and $\lambda, \mu \in \mathfrak{d}^{-1}$, it follows that $a^*, b^* \in \mathbb{Z}$ and that $\lambda^*, \mu^* \in \mathfrak{d}^{-1}$ (since $\mathfrak{d}^{-1}$ is a fractional ideal). Moreover, $(a^*, b^*, \lambda^*, \mu^*)$ runs over $\mathbb{Z} \times \mathbb{Z} \times \mathfrak{d}^{-1} \times \mathfrak{d}^{-1}$ as (a, b, λ, μ) does.

So (under the assumption of absolute convergence), to complete the proof that H_m transforms like a Hilbert modular form of weight k we must check that $h(a^*, b^*, \lambda^*, \mu^*) = h(a, b, \lambda, \mu)$ and that $\lambda^*(\lambda^*)' - a\mu^* \gg 0$.

First, a (lengthy) computation shows that the hypermatrix A^* defined by $a^*, b^*, \lambda^*, \mu^*$ is obtained from the hypermatrix A defined by a, b, λ, μ by

$$A^* = (M^T, (M')^T, (M'')^T) \cdot A \tag{5.18}$$

so it follows from (5.9) that the two hypermatrices have the same hyperdeterminant.

Next, another computation shows that

$$\lambda^*\lambda^{*\prime} - a^*\mu^* = (\lambda\lambda' - a\mu)\alpha^2 + (\lambda\mu' - ab + \lambda'\mu'' - \lambda''\mu)\alpha\gamma + (\mu'\mu'' - b\lambda'')\gamma^2 \tag{5.19}$$

The right-hand side is a quadratic form in α and γ whose discriminant is

$$(ab - \lambda\mu' - \lambda'\mu'' + \lambda''\mu)^2 - 4(\lambda\lambda' - a\mu)(\mu'\mu'' - b\lambda'') = h(a, b, \lambda, \mu) < 0 \tag{5.20}$$

Therefore this quadratic form always has the same sign. Since it is positive when $\alpha = 1$ and $\gamma = 0$, it follows that $\lambda^*\lambda^{*\prime} - a^*\mu^*$ is positive as well. If we conjugate the above identity once or twice, the same argument will show that the conjugates of $\lambda^*\lambda^{*\prime} - a^*\mu^*$ are positive.

5.6 Non-vanishing

In this section we will show that the conditions $h(a, b, \lambda, \mu) < 0$ and $\lambda\lambda' - a\mu \gg 0$ guarantee that the expression

$$az_1z_2z_3 + \lambda' z_1z_2 + \lambda z_1z_3 + \lambda'' z_2z_3 + \mu z_1 + \mu' z_2 + \mu'' z_3 + b$$

does not vanish in $\mathbb{H}^3$.

Suppose for the sake of contradiction that it does vanish for some values of $a, b, \lambda, \mu, z_1, z_2, z_3$ satisfying the above conditions. Then we obtain

$$z_1(az_2z_3 + \lambda' z_2 + \lambda z_3 + \mu) = -(\lambda'' z_2z_3 + \mu' z_2 + \mu'' z_3 + b) \tag{5.21}$$

If one side of (5.21) is zero then the other side is zero as well, and the left-hand side then implies that

$$z_2(az_3 + \lambda') = -(\lambda z_3 + \mu)$$

or

$$z_2 = -\frac{\lambda z_3 + \mu}{az_3 + \lambda'} \tag{5.22}$$

Since z_2 and z_3 both have positive imaginary parts, this is only possible if $\lambda\lambda' - a\mu < 0$, which is contrary to the hypotheses on a, b, λ, and μ.

So we may assume that both sides of (5.21) are non-zero. Therefore

$$z_1 = -\frac{\lambda'' z_2z_3 + \mu' z_2 + \mu'' z_3 + b}{az_2z_3 + \lambda' z_2 + \lambda z_3 + \mu} \tag{5.23}$$

If we define

$$w = \frac{\lambda'' z_2z_3 + \mu' z_2 + \mu'' z_3 + b}{az_2z_3 + \lambda' z_2 + \lambda z_3 + \mu} \in \mathbb{C} \tag{5.24}$$

then it is enough to show that $\Im w > 0$, for then we would have $\Im z_1 < 0$, a contradiction.

And if we write $z_j = x_j + iy_j$ for $j = 2, 3$, then a computation shows that

$$\Im w = P(x_2)y_3 + Q(x_3)y_2 + (\lambda\lambda'' - a\mu'')y_2y_3^2 + (\lambda'\lambda'' - a\mu')y_2^2y_3 \tag{5.25}$$

where

$$P(x_2) = (\lambda'\lambda'' - a\mu')x_2^2 + (\lambda''\mu - ab - \lambda\mu' + \lambda'\mu'')x_2 + (\mu\mu'' - b\lambda) \tag{5.26}$$

and

$$Q(x_3) = (\lambda\lambda'' - a\mu'')x_3^2 + (\lambda\mu' - ab - \lambda'\mu'' + \lambda''\mu)x_3 + (\mu\mu' - b\lambda') \tag{5.27}$$

$P(x)$ and $Q(x)$ are quadratic polynomials, both of whose discriminants are equal to $h(a,b,\lambda,\mu)$. Therefore $P(x)$ and $Q(x)$ do not change sign. And since

$$P(0) = \mu\mu'' - b\lambda > 0$$

and

$$Q(0) = \mu\mu' - b\lambda' > 0$$

it follows that $P(x) > 0$ and $Q(x) > 0$ for all x. Since $y_2, y_3 > 0$, it follows that every term on the right-hand side of (5.25) is positive, so $\Im w > 0$.

5.7 Absolute convergence

The goal of this section is to show that if $m > 0$ then the series in the definition of $H_m(z_1, z_2, z_3)$ converges absolutely for sufficiently large k.

To do so, observe that if (a, b, λ, μ) satisfies the conditions in the sum defining H_m, then so does $(-a, -b, -\lambda, -\mu)$. Therefore

$$\sum_{\substack{a,b\in\mathbb{Z} \\ \lambda,\mu\in\mathfrak{d}^{-1} \\ h(a,b,\lambda,\mu)=-m/\Delta^2 \\ \lambda\lambda'-a\mu\gg 0}} |az_1z_2z_3 + \lambda'z_1z_2 + \lambda z_1z_3 + \lambda''z_2z_3 + \mu z_1 + \mu'z_2 + \mu''z_3 + b|^{-k}$$

$$= \sum_{\substack{b\in\mathbb{Z} \\ \lambda,\mu\in\mathfrak{d}^{-1} \\ h(0,b,\lambda,\mu)=-m/\Delta^2 \\ \lambda\lambda'\gg 0}} |\lambda'z_1z_2 + \lambda z_1z_3 + \lambda''z_2z_3 + \mu z_1 + \mu'z_2 + \mu''z_3 + b|^{-k}$$

$$+2\sum_{a=1}^{\infty} \sum_{\substack{b\in\mathbb{Z} \\ \lambda,\mu\in\mathfrak{d}^{-1} \\ h(a,b,\lambda,\mu)=-m/\Delta^2 \\ \lambda\lambda'-a\mu\gg 0}} |az_1z_2z_3 + \lambda'z_1z_2 + \lambda z_1z_3 + \lambda''z_2z_3 + \mu z_1 + \mu'z_2 + \mu''z_3 + b|^{-k}$$

The summand $az_1z_2z_3 + \lambda' z_1z_2 + \lambda z_1z_3 + \lambda'' z_2z_3 + \mu z_1 + \mu' z_2 + \mu'' z_3 + b$ can be expressed in the following three ways (as in the previous section):

$$\Big(az_3 + \lambda'\Big)\Big(z_2 + \frac{\lambda z_3 + \mu}{az_3 + \lambda'}\Big)\Big(z_1 + \frac{\lambda'' z_2z_3 + \mu' z_2 + \mu'' z_3 + b}{az_2z_3 + \lambda' z_2 + \lambda z_3 + \mu}\Big) \tag{5.28}$$

$$\Big(az_2 + \lambda\Big)\Big(z_3 + \frac{\lambda' z_2 + \mu}{az_2 + \lambda}\Big)\Big(z_1 + \frac{\lambda'' z_2z_3 + \mu' z_2 + \mu'' z_3 + b}{az_2z_3 + \lambda' z_2 + \lambda z_3 + \mu}\Big) \tag{5.29}$$

$$\Big(az_1 + \lambda''\Big)\Big(z_2 + \frac{\lambda z_1 + \mu''}{az_1 + \lambda''}\Big)\Big(z_3 + \frac{\lambda' z_1z_2 + \mu z_1 + \mu' z_2 + b}{az_1z_2 + \lambda z_1 + \lambda'' z_2 + \mu''}\Big) \tag{5.30}$$

Now fix $z_1, z_2, z_3 \in \mathbb{H}$, and let

$$\delta = \min\{\Im z_1, \Im z_2, \Im z_3\} > 0$$

Also, by translating we can assume that $|\Re z_j| \leq \frac{1}{2}$ for $j = 1, 2, 3$.

First assume that $a > 0$.

Using (5.28) and the estimates

$$\Im(az_3 + \lambda') = \Im(az_3) \geq a\delta \tag{5.31}$$

$$\Big|z_2 + \frac{\lambda z_3 + \mu}{az_3 + \lambda'}\Big| \geq \Im\Big[z_2 + \frac{\lambda z_3 + \mu}{az_3 + \lambda'}\Big] \geq \delta + \frac{(\lambda\lambda' - a\mu)\delta}{|az_3 + \lambda'|^2} \tag{5.32}$$

$$\begin{aligned}\Big|z_1 + \frac{\lambda'' z_2z_3 + \mu' z_2 + \mu'' z_3 + b}{az_2z_3 + \lambda' z_2 + \lambda z_3 + \mu}\Big| &\geq \delta + \Im\Big[\frac{\lambda'' z_2z_3 + \mu' z_2 + \mu'' z_3 + b}{az_2z_3 + \lambda' z_2 + \lambda z_3 + \mu}\Big] \\ &\geq \delta + (\lambda\lambda'' - a\mu'')\delta^3 + (\lambda'\lambda'' - a\mu')\delta^3\end{aligned} \tag{5.33}$$

(with the last inequality using the result of the previous section) we obtain

$$|az_1z_2z_3 + \lambda' z_1z_2 + \lambda z_1z_3 + \lambda'' z_2z_3 + \mu z_1 + \mu' z_2 + \mu'' z_3 + b|$$

$$\gg_\delta \Big(a + |\Re(az_3 + \lambda')|\Big)\Big(1 + \frac{\lambda\lambda' - a\mu}{|az_3 + \lambda'|^2}\Big)\Big(1 + \lambda'\lambda'' - a\mu' + \lambda'\lambda'' - a\mu'\Big)$$

Similarly, using (5.29) we get

$$|az_1z_2z_3 + \lambda' z_1z_2 + \lambda z_1z_3 + \lambda'' z_2z_3 + \mu z_1 + \mu' z_2 + \mu'' z_3 + b|$$

$$\gg_\delta \Big(a + |\Re(az_2 + \lambda)|\Big)\Big(1 + \frac{\lambda\lambda' - a\mu}{|az_2 + \lambda|^2}\Big)\Big(1 + \lambda'\lambda'' - a\mu' + \lambda'\lambda'' - a\mu'\Big)$$

and using (5.30) we obtain

$$|az_1z_2z_3 + \lambda' z_1z_2 + \lambda z_1z_3 + \lambda'' z_2z_3 + \mu z_1 + \mu' z_2 + \mu'' z_3 + b|$$

$$\gg_\delta \Big(a + |\Re(az_1 + \lambda'')|\Big)\Big(1 + \frac{\lambda\lambda'' - a\mu''}{|az_1 + \lambda''|^2}\Big)\Big(1 + \lambda'\lambda'' - a\mu' + \lambda\lambda' - a\mu\Big)$$

Now (still assuming $a > 0$) for $R = 1, 2, \dots$ define

$$N(R) = \#\{(a, b, \lambda, \mu) : |az_1z_2z_3 + \lambda' z_1z_2 + \lambda z_1z_3 + \lambda'' z_2z_3 + \mu z_1 + \mu' z_2 + \mu'' z_3 + b| \le R\}$$

From the estimates above and the fact that $\lambda\lambda' - a\mu \gg 0$ we see that there is constant C such that if

$$|az_1z_2z_3 + \lambda' z_1z_2 + \lambda z_1z_3 + \lambda'' z_2z_3 + \mu z_1 + \mu' z_2 + \mu'' z_3 + b| \le R$$

then necessarily

$$1 \le a \le R \qquad \max\{|\lambda|, |\lambda'|, |\lambda''|\} \le CR \qquad \max\{|\mu|, |\mu'|, |\mu''|\} \le CR^2 \tag{5.34}$$

from which it follows that

$$N(R) \ll R^{10}$$

Therefore

$$2\sum_{a=1}^{\infty} \sum_{\substack{b\in\mathbb{Z} \\ \lambda,\mu\in\mathfrak{d}^{-1} \\ h(a,b,\lambda,\mu)=-m/\Delta^2 \\ \lambda\lambda'-a\mu\gg 0}} |az_1z_2z_3 + \lambda' z_1z_2 + \lambda z_1z_3 + \lambda'' z_2z_3 + \mu z_1 + \mu' z_2 + \mu'' z_3 + b|^{-k}$$

$$\ll \sum_{R=1}^{\infty} \frac{N(R+1) - N(R)}{R^k} \ll \sum_{R=1}^{\infty} \frac{N(R)}{R^k} \ll \sum_{R=1}^{\infty} R^{10-k}$$

which converges for $k \ge 12$.

Finally, let us deal with the sum for $a = 0$. In this case, since $h(a, b, \lambda, \mu) < 0$ it follows that $\lambda \neq 0$ (hence its conjugates are non-zero as well). And now using (5.28)-(5.30) we get

$$\begin{aligned} &|\lambda' z_1 z_2 + \lambda z_1 z_3 + \lambda'' z_2 z_3 + \mu z_1 + \mu' z_2 + \mu'' z_3 + b| \\ &\gg_\delta |\lambda'|\Big(1 + \Big|\frac{\lambda}{\lambda'} + \frac{\mu}{\lambda'}\Big|\Big) \\ &\gg_\delta |\lambda|\Big(1 + \Big|\frac{\lambda'}{\lambda} + \frac{\mu}{\lambda}\Big|\Big) \\ &\gg_\delta |\lambda''|\Big(1 + \Big|\frac{\lambda}{\lambda''} + \frac{\mu''}{\lambda''}\Big|\Big) \end{aligned}$$

From this it follows that if

$$|\lambda' z_1 z_2 + \lambda z_1 z_3 + \lambda'' z_2 z_3 + \mu z_1 + \mu' z_2 + \mu'' z_3 + b| \leq R$$

then there is a constant C such that

$$\max\{|\lambda|, |\lambda'|, |\lambda''|\} \leq CR \qquad \max\{|\mu|, |\mu'|, |\mu''|\} \leq CR^2$$

hence

$$N(R) \ll R^9$$

hence once again $k \geq 12$ guarantees that

$$\sum_{\substack{b \in \mathbb{Z} \\ \lambda, \mu \in \mathfrak{d}^{-1} \\ h(0,b,\lambda,\mu) = -m/\Delta^2 \\ \lambda\lambda' \gg 0}} |\lambda' z_1 z_2 + \lambda z_1 z_3 + \lambda'' z_2 z_3 + \mu z_1 + \mu' z_2 + \mu'' z_3 + b|^{-k} < \infty$$

Thus we have shown that $H_m(z_1, z_2, z_3)$ converges absolutely if $k \geq 12$.

5.8 The case $m = 0$

The definition of H_m is slightly different when $m = 0$: we now define

$$H_0(z_1, z_2, z_3) = \sideset{}{'}\sum_{\substack{a,b \in \mathbb{Z} \\ \lambda, \mu \in \mathfrak{d}^{-1} \\ h(a,b,\lambda,\mu) = 0 \\ \lambda\lambda' - a\mu = 0 \\ \mu\mu'' - b\lambda = 0}} (a z_1 z_2 z_3 + \lambda' z_1 z_2 + \lambda z_1 z_3 + \lambda'' z_2 z_3 + \mu z_1 + \mu' z_2 + \mu'' z_3 + b)^{-k}$$

The prime on the summation means that the quadruple $(0,0,0,0)$ should be omitted.

The condition $\mu\mu'' - b\lambda = 0$ may seem new, but if we recall that

$$h(a,b,\lambda,\mu) = (ab - \lambda\mu' - \lambda'\mu'' + \lambda''\mu)^2 - 4(\lambda\lambda' - a\mu)(\mu'\mu'' - b\lambda'')$$

we see that in the case $m > 0$, $h(a,b,\lambda,\mu) < 0$ and $\lambda\lambda' - a\mu \gg 0$ together imply that $\mu\mu'' - b\lambda \gg 0$ as well. So in the case $m = 0$ it is natural (and useful) to include this condition.

Let us first see what the conditions $h(a,b,\lambda,\mu) = \lambda\lambda' - a\mu = \mu\mu'' - b\lambda = 0$ imply. First, since

$$h(a,b,\lambda,\mu) = (ab - \lambda\mu' - \lambda'\mu'' + \lambda''\mu)^2 - 4(\lambda\lambda' - a\mu)(\mu'\mu'' - b\lambda'')$$

it follows that

$$ab - \lambda\mu' - \lambda'\mu'' + \lambda''\mu = 0 \tag{5.35}$$

and conjugating this expression yields

$$ab - \lambda'\mu'' - \lambda''\mu + \lambda\mu' = 0 \tag{5.36}$$

Adding these two expressions and dividing by 2, we obtain

$$ab - \lambda'\mu'' = 0 \tag{5.37}$$

which in turn implies that

$$\lambda''\mu - \lambda\mu' = 0 \tag{5.38}$$

So we now know that

$$\lambda\lambda' = a\mu \qquad \lambda'\mu'' = ab \qquad \lambda''\mu = \lambda\mu' \qquad b\lambda = \mu\mu'' \tag{5.39}$$

Multiplying the first condition by λ'', we get

$$N(\lambda) = a\mu\lambda'' = a^2 b \in \mathbb{Z} \tag{5.40}$$

and summing the three conjugates of the first condition yields

$$\lambda\lambda' + \lambda\lambda'' + \lambda'\lambda'' = a\mathrm{tr}(\mu) \in \mathbb{Z} \tag{5.41}$$

Therefore the coefficients of the minimal polynomial for λ are integers, so λ is an algebraic integer. Similarly, we can use the last condition to conclude that μ is an algebraic integer. So the definition of H_0 can be rewritten as

$$H_0(z_1, z_2, z_3) = \sideset{}{'}\sum_{\substack{a,b\in\mathbb{Z},\lambda,\mu\in\mathcal{O}_K \\ \lambda\lambda'=a\mu,\ \mu\mu''=b\lambda \\ \lambda'\mu''=ab,\ \lambda''\mu=\lambda\mu'}} (az_1z_2z_3 + \lambda' z_1z_2 + \lambda z_1z_3 + \lambda'' z_2z_3 + \mu z_1 + \mu' z_2 + \mu'' z_3 + b)^{-k}$$

Now we will show that H_0 is a multiple of the Hecke-Eisenstein series for K. First let us briefly recall the definition of these series, following the exposition in Zagier's paper:

If K is a totally real cubic field and $\mathfrak{c}$ is an ideal class of K, define

$$E_k(z_1, z_2, z_3; \mathfrak{c}) = 2N(\mathfrak{a})^k \sum_{(\varepsilon,\eta)\in(\mathfrak{a}\times\mathfrak{a}\backslash\{(0,0)\})/\mathcal{O}_k^\times} [(\varepsilon z_1 + \eta)(\varepsilon' z_2 + \eta')(\varepsilon'' z_3 + \eta'')]^{-k} \tag{5.42}$$

where $\mathfrak{a}$ is a fractional ideal contained in $\mathfrak{c}$. The Hecke-Eisenstein series of weight k for K is then

$$E_k(z_1, z_2, z_3) = \sum_{\mathfrak{c}} E_k(z_1, z_2, z_3; \mathfrak{c}) \tag{5.43}$$

with the sum running over all ideal classes in the class group. For the proof that H_0 is a multiple of E_k, it is useful to decompose E_k is a slightly different form: define

$$E_k^*(z_1, z_2, z_3; \mathfrak{c}) = 2N(\mathfrak{a})^k \sum_{\gcd(\varepsilon,\eta)=\mathfrak{a}} [(\varepsilon z_1 + \eta)(\varepsilon' z_2 + \eta')(\varepsilon'' z_3 + \eta'')]^{-k} \tag{5.44}$$

where $\mathfrak{a}$ is again a fractional ideal in $\mathfrak{c}$, and the sum is over non-associated $\varepsilon, \eta \in K$ such that the greatest common divisor $\gcd(\varepsilon, \eta)$ of ε and η is equal to $\mathfrak{a}$. Using E_k^*, we can write

$$E_k(z_1, z_2, z_3; \mathfrak{c}) = 2N(\mathfrak{a})^k \sum_{\mathfrak{b}} \sum_{\gcd(\varepsilon,\eta)=\mathfrak{a}\mathfrak{b}} [(\varepsilon z_1 + \eta)(\varepsilon' z_2 + \eta')(\varepsilon'' z_3 + \eta'')]^{-k}$$

$$= 2N(\mathfrak{a})^k \sum_{\mathfrak{b}} [2N(\mathfrak{a}\mathfrak{b})^k]^{-1} E_k^*(z_1, z_2, z_3; [\mathfrak{a}\mathfrak{b}]) = \sum_{\mathfrak{b}} N(\mathfrak{b})^{-k} E_k^*(z_1, z_2, z_3; \mathfrak{c}[\mathfrak{b}])$$

where $\mathfrak{b}$ runs over all non-zero integral ideals. Hence

$$E_k(z_1, z_2, z_3) = \sum_{\mathfrak{c}} E_k(z_1, z_2, z_3; \mathfrak{c})$$

$$= \sum_{\mathfrak{c}} \sum_{\mathfrak{b}} N(\mathfrak{b})^{-k} E_k^*(z_1, z_2, z_3; \mathfrak{c}[\mathfrak{b}]) = \zeta_K(k) \sum_{\mathfrak{c}} E_k^*(z_1, z_2, z_3; \mathfrak{c})$$

where $\zeta_K(s)$ is the Dedekind zeta function for K.

On the other hand,

$$H_0(z_1,z_2,z_3) = \sum_{\substack{a,b\in\mathbb{Z},\lambda,\mu\in\mathcal{O}_K\\ \lambda\lambda'=a\mu,\ \mu\mu''=b\lambda\\ \lambda'\mu''=ab,\ \lambda''\mu=\lambda\mu'}}{}^{'} (az_1z_2z_3+\lambda' z_1z_2+\lambda z_1z_3+\lambda'' z_2z_3+\mu z_1+\mu' z_2+\mu'' z_3+b)^{-k}$$

$$= \sum_{r=1}^{\infty} r^{-k} \sum_{\substack{a,b,\lambda,\mu \text{ primitive}\\ \lambda\lambda'=a\mu,\ \mu\mu''=b\lambda\\ \lambda'\mu''=ab,\ \lambda''\mu=\lambda\mu'}} (az_1z_2z_3+\lambda' z_1z_2+\lambda z_1z_3+\lambda'' z_2z_3+\mu z_1+\mu' z_2+\mu'' z_3+b)^{-k}$$

$$= \zeta(k)H_0^*(z_1,z_2,z_3)$$

where primitive means that no integer divides all of a,b,λ,μ. If we then define

$$H_0^*(z_1,z_2,z_3;\mathfrak{c}) = \sum_{\substack{a,b,\lambda,\mu \text{ primitive}\\ \gcd(a,\lambda'')\in\mathfrak{c}\\ \lambda\lambda'=a\mu,\ \mu\mu''=b\lambda\\ \lambda'\mu''=ab,\ \lambda''\mu=\lambda\mu'}} az_1z_2z_3+\lambda' z_1z_2+\lambda z_1z_3+\lambda'' z_2z_3+\mu z_1+\mu' z_2+\mu'' z_3+b)^{-k}$$

then

$$H_0(z_1,z_2,z_3) = \zeta(k)H_0^*(z_1,z_2,z_3) = \zeta(k)\sum_{\mathfrak{c}} H_0^*(z_1,z_2,z_3;\mathfrak{c})$$

Hence if we can show that

$$H_0^*(z_1,z_2,z_3;\mathfrak{c}) = E_k^*(z_1,z_2,z_3;\mathfrak{c})$$

then it will follow that

$$H_0(z_1,z_2,z_3) = \zeta(k)\sum_{\mathfrak{c}} E_k^*(z_1,z_2,z_3;\mathfrak{c}) = \frac{\zeta(k)}{\zeta_K(k)}E_k(z_1,z_2,z_3)$$

In order to show that $H_0^*(z_1,z_2,z_3;\mathfrak{c}) = E_k^*(z_1,z_2,z_3;\mathfrak{c})$, define S to be the set of all $(a,b,\lambda,\mu)\in\mathbb{Z}^2\times\mathcal{O}_K^2$ satisfying the conditions in the sum defining $H_0^*(z_1,z_2,z_3;\mathfrak{c})$, and define

$$T = \{(\varepsilon,\eta)\in(\mathfrak{a}\times\mathfrak{a}\setminus\{(0,0)\})/\mathcal{O}_K^\times : \gcd(\varepsilon,\eta)=\mathfrak{a}\} \tag{5.45}$$

Define a map $\Phi : S\to T$ as follows: given $(a,b,\lambda,\mu)\in S$, if $b\neq 0$ then at least one of the fractions

$$\frac{a}{\lambda''} = \frac{\lambda'}{\mu'} = \frac{\mu}{b} \tag{5.46}$$

is well-defined. Since $\gcd(a, \lambda'') \in \mathfrak{c}$, it follows that we can represent this quotient as a fraction $\frac{x}{y}$ with $x, y \in \mathfrak{a}$.

Since $\gcd(x, y)$ is contained in $\mathfrak{a}$, which is an element of $\mathfrak{c}$, it follows that $\gcd(x, y) = (\omega)\mathfrak{a}$ for some $\omega \in K^{\times}$. Both x and y are divisible by ω, hence $\varepsilon = \frac{x}{\omega}$ and $\eta = \frac{y}{\omega}$ are in $\mathfrak{a}$ and have gcd $\mathfrak{a}$.

Moreover, the conditions $\frac{\varepsilon}{\eta} = \frac{a}{\lambda''} = \frac{\mu}{b}$ and $\gcd(\varepsilon, \eta) = \mathfrak{a}$ define ε and η up to multiplication by a unit, so define $\Phi(a, b, \lambda, \mu)$ to be the class of (ε, η) in T.

If $b = 0$, then $\lambda = \mu = 0$ as well, and then the primitivity condition implies that $a = \pm 1$. Also $\lambda = 0$ implies that $\gcd(a, \lambda'')$ is principal, i.e. $\mathfrak{c}$ is the identity class. So define $\Phi(\pm 1, 0, 0, 0) = (\varepsilon, 0)$, where $(\varepsilon) = \mathfrak{a}$.

Now let us show that Φ is surjective:

Suppose that the ordered pair (ε, η) represents a class in T. We have just seen that if $\eta = 0$ then $(\varepsilon, 0)$ is the image under Φ of an element of S, so assume $\eta \neq 0$. Choose a non-zero integer b_1 such that

$$\lambda_1 = b_1 \frac{\varepsilon \varepsilon''}{\eta \eta''} \in \mathcal{O}_K \tag{5.47}$$

and

$$\mu_1 = b_1 \frac{\varepsilon}{\eta} \in \mathcal{O}_K \tag{5.48}$$

If we define a_1 by

$$a_1 b_1 = \lambda_1 \mu_1' = b_1^2 \frac{N(\varepsilon)}{N(\eta)} \tag{5.49}$$

then $a_1 \in \mathbb{Q}$. Choose a non-zero integer m such that $a_2 = m a_1 \in \mathbb{Z}$, and define $b_2 = m b_1$, $\lambda_2 = m \lambda_1$, and $\mu_2 = m \mu_1$. Then $(a_2, b_2, \lambda_2, \mu_2) \in \mathbb{Z}^2 \times \mathcal{O}_K^2$ with

$$\frac{a_2}{\lambda_2''} = \frac{a_1}{\lambda_1''} = \frac{\mu_1}{b_1} = \frac{\varepsilon}{\eta}$$

Also

$$N(\eta)\lambda_2'' \in \mathfrak{a} \qquad N(\eta) a_2 \in \mathfrak{a}$$

so $\gcd(a_2, \lambda_2'') \in \mathfrak{c}$.

Therefore we can factor out any common integer divisor of $a_2, b_2, \lambda_2, \mu_2$ to obtain an element (a, b, λ, μ) of S with $\Phi(a, b, \lambda, \mu) = (\varepsilon, \eta)$. So Φ is surjective.

Next, Φ is $2-1$: we will show that $\Phi(a_1, b_1, \lambda_1, \mu_1) = \Phi(a_2, b_2, \lambda_2, \mu_2)$ if and only if $(a_1, b_1, \lambda_1, \mu_1) = \pm(a_2, b_2, \lambda_2, \mu_2)$.

Suppose that $\Phi(a_1, b_1, \lambda_1, \mu_1) = \Phi(a_2, b_2, \lambda_2, \mu_2)$. Once again we can assume that $b_1, b_2 \neq 0$. Then

$$\frac{\mu_1}{b_1} = \frac{\mu_2}{b_2} \qquad \frac{a_1}{\lambda_1''} = \frac{a_2}{\lambda_2''} \qquad \frac{\lambda_1'}{\mu_1'} = \frac{\lambda_2'}{\mu_2'}$$

Let b_0 be the gcd of b_1 and b_2, and define

$$\mu_0 = \frac{b_0}{b_1}\mu_1 \qquad \lambda_0 = \frac{b_0}{b_1}\lambda_1 \qquad a_0 = \frac{b_0}{b_1}a_1 \tag{5.50}$$

There exists $r, s \in \mathbb{Z}$ such that $b_0 = rb_1 + sb_2$, hence

$$\mu_0 = \frac{rb_1 + sb_2}{b_1}\mu_1 = r\mu_1 + s\frac{b_2}{b_1}\mu_1 = r\mu_1 + sb_2\frac{\mu_2}{b_2} = r\mu_1 + s\mu_2 \in \mathcal{O}_K$$

and similarly

$$\lambda_0' = \frac{rb_1 + sb_2}{b_1}\lambda_1' = r\lambda_1' + s\lambda_2' \in \mathcal{O}_K$$

and

$$a_0 = ra_1 + sa_2 \in \mathbb{Z}$$

Therefore we obtain a new quadruple $(a_0, b_0, \lambda_0, \mu_0)$ with

$$(a_1, b_1, \lambda_1, \mu_1) = \frac{b_1}{b_0}(a_0, b_0, \lambda_0, \mu_0) \tag{5.51}$$

Since $\frac{b_1}{b_0} \in \mathbb{Z}$, the primitivity condition on $(a_1, b_1, \lambda_1, \mu_1)$ implies that $\frac{b_1}{b_0} = \pm 1$.

Similarly we obtain $\frac{b_2}{b_0} = \pm 1$, so $(a_1, b_1, \lambda_1, \mu_1) = \pm(a_2, b_2, \lambda_2, \mu_2)$.

So we have shown that Φ is $2-1$ and surjective. And if $\Phi(a, b, \lambda, \mu) = (\varepsilon, \eta)$, then

$$(\varepsilon z_1 + \eta)(\varepsilon' z_2 + \eta')(\varepsilon'' z_3 + \eta'')$$

$$= n(az_1z_2z_3 + \lambda' z_1z_2 + \lambda z_1z_3 + \lambda'' z_2z_3 + \mu z_1 + \mu' z_2 + \mu'' z_3 + b)$$

with $n \in \mathbb{Q}^\times$. The primitivity of (a, b, λ, μ) implies that $n \in \mathbb{Z}$, and then $|n|$ is the largest integer dividing

$$\gcd(\varepsilon\varepsilon'\varepsilon'', \varepsilon\varepsilon'\eta'', \varepsilon\varepsilon''\eta', \varepsilon'\varepsilon''\eta, \varepsilon\eta'\eta'', \varepsilon'\eta\eta'', \varepsilon''\eta\eta', \eta\eta'\eta'')$$

$$= \gcd(\varepsilon(\varepsilon'\varepsilon'', \varepsilon'\eta'', \varepsilon''\eta', \eta'\eta''), \eta(\varepsilon'\varepsilon'', \varepsilon'\eta'', \varepsilon''\eta', \eta'\eta''))$$

$$\gcd(\varepsilon, \eta) \gcd(\varepsilon'(\varepsilon'', \eta''), \eta'(\varepsilon'', \eta''))$$

$$= \gcd(\varepsilon, \eta) \gcd(\varepsilon', \eta') \gcd(\varepsilon'', \eta'') = \mathfrak{a}\mathfrak{a}'\mathfrak{a}'' = N(\mathfrak{a})$$

Therefore $n = \pm N(\mathfrak{a})$. Combining all of the above, we obtain

$$H_0^*(z_1, z_2, z_3; \mathfrak{c}) = E_k^*(z_1, z_2, z_3; \mathfrak{c})$$

which completes the proof.

CHAPTER 6

Modular Forms for Cubic Fields: One variable

In this chapter we construct a family of one-variable modular forms which should be related to the forms of the previous chapter in a manner analogous to the way the modular forms $f_{k,\Delta}$ in section 4.6 are related to the Hilbert modular forms ω_m. We also provide two different expressions for the Fourier expansions of these forms.

6.1 Definitions

Let $\Delta > 0$ be a cubic discriminant, and $k \geq 8$ an even integer. Define

$$h_{k,\Delta}(z) = \sum_{\substack{a,b,c,d \in \mathbb{Z} \\ \mathrm{disc}(a,b,c,d)=\Delta}} (az^3 + bz^2 + cz + d)^{-k} \tag{6.1}$$

Here z lies in the upper half plane $\mathbb{H}$ and

$$\mathrm{disc}(a,b,c,d) = -27a^2d^2 + 18abcd - 4ac^3 - 4b^3d + b^2c^2$$

is the discriminant of the cubic polynomial $aX^3 + bX^2 + cX + d$.

Recall that if $\Delta > 0$ then the polynomial $aX^3 + bX^2 + cX + d$ has distinct real roots, hence each term in the series is non-zero. Therefore, if the series converges absolutely then $f_{k,\Delta}$ is a holomorphic function on $\mathbb{H}$.

If $\gamma = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ is an element of $\mathrm{SL}_2(\mathbb{Z})$, then the map $z \mapsto \gamma z$ sends $az^3 + bz^2 + cz + d$ to

$$(rz + s)^{-3}(a^*z^3 + b^*z^2 + c^*z + d^*)$$

with

$$a^* = ap^3 + bp^2r + cpr^2 + dr^3 \tag{6.2}$$

$$b^* = 3ap^2q + b(p^2s + 2pqr) + c(2prs + qr^2) + 3dr^2s \tag{6.3}$$

$$c^* = 3apq^2 + b(2pqs + q^2r) + c(ps^2 + 2qrs) + 3drs^2 \tag{6.4}$$

$$d^* = aq^3 + bq^2s + cqs^2 + ds^3 \tag{6.5}$$

Since the discriminant is an invariant of the action of $\mathrm{SL}_2(\mathbb{Z})$ on binary cubic forms, it follows that if the series in (6.1) converges absolutely, then $h_{k,\Delta}(z)$ transforms like a modular form of weight $3k$.

6.2 Absolute convergence

In this section we will prove that the series (6.1) converges absolutely for even integers $k \geq 8$. First observe that if $\gamma = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ is an element of $\mathrm{SL}_2(\mathbb{Z})$, then

$$\sum_{\substack{a,b,c,d\in\mathbb{Z} \\ \mathrm{disc}(a,b,c,d)=\Delta}} |a(\gamma z)^3 + b(\gamma z)^2 + c(\gamma z) + d|^{-k} = |rz+s|^{3k} \sum_{\substack{a,b,c,d\in\mathbb{Z} \\ \mathrm{disc}(a,b,c,d)=\Delta}} |az^3 + bz^2 + cz + d|^{-k}$$

by Tonelli's theorem. Therefore we may assume that z lies in the standard fundamental domain for the modular group, i.e. $z = x + iy$ with $|z| \geq 1$ and $|x| \leq \frac{1}{2}$.

Next, note that if a term (a, b, c, d) appears in the sum, then so does $(-a, -b, -c, -d)$. Therefore

$$\sum_{\substack{a,b,c,d\in\mathbb{Z} \\ \mathrm{disc}(a,b,c,d)=\Delta}} |az^3 + bz^2 + cz + d|^{-k} = \sum_{\substack{b,c,d\in\mathbb{Z} \\ \mathrm{disc}(b,c,d)=\Delta}} |bz^2 + cz + d|^{-k}$$

$$+2\sum_{a=1}^{\infty} \sum_{\substack{a,b,c,d\in\mathbb{Z} \\ \mathrm{disc}(a,b,c,d)=\Delta}} |az^3 + bz^2 + cz + d|^{-k}$$

The first sum is essentially the modular form $f_k(\Delta, z)$ defined in section 4.6, so is known to converge for $k \geq 4$.

For the second sum, fix $a \geq 1$ and consider

$$\sum_{\substack{b,c,d\in\mathbb{Z} \\ \mathrm{disc}(a,b,c,d)=\Delta}} |az^3 + bz^2 + cz + d|^{-k}$$

The discriminant condition is difficult to work with, so we will use only the fact that $\operatorname{disc}(a, b, c, d) > 0$. Therefore let $N_a(R)$ denote the number of triples (b, c, d) such that $\operatorname{disc}(a, b, c, d) > 0$ and $|az^3 + bz^2 + cz + d| \leq aR$.

Write

$$|az^3 + bz^2 + cz + d| = a|z - r_1| \cdot |z - r_2| \cdot |z - r_3|$$

$$= a[(x - r_1)^2 + y^2]^{\frac{1}{2}}[(x - r_2)^2 + y^2]^{\frac{1}{2}}[(x - r_3)^2 + y^2]^{\frac{1}{2}}$$

where the roots r_j are real and distinct.

Each of the three terms in the product is bounded below by y, and $y \geq \frac{\sqrt{3}}{2}$ since z lies in the fundamental domain. Moreover, if $R \geq 1$ and $|r_j| > R$, then

$$|x - r_j| \geq |r_j| - |x| \geq \frac{1}{2}|r_j|$$

since $|x| \leq \frac{1}{2}$. Therefore $N_a(R)$ is bounded above by a constant multiple of the number of triples (b, c, d) such that all three roots r_j of (a, b, c, d) satisfy $|r_j| \leq R$.

Consider such a triple, and let $p(t) = at^3 + bt^2 + ct + d$. Then by Rolle's theorem, $p'(t) = 3at^2 + 2bt + c$ has two distinct roots, which also lie in the interval $[-R, R]$. By the quadratic formula, these roots are

$$\frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$$

Since the roots are real and distinct, the discriminant $b^2 - 3ac$ is positive, which implies that

$$c < \frac{b^2}{3a}$$

Moreover, by adding the two roots, we obtain

$$\frac{2|b|}{3a} = \left|\frac{-b + \sqrt{b^2 - 3ac}}{3a} + \frac{-b - \sqrt{b^2 - 3ac}}{3a}\right|$$

$$\leq \left|\frac{-b + \sqrt{b^2 - 3ac}}{3a}\right| + \left|\frac{-b - \sqrt{b^2 - 3ac}}{3a}\right| \leq 2R$$

hence

$$|b| \leq 3aR$$

On the other hand, subtracting the roots yields

$$\frac{2\sqrt{b^2-3ac}}{3a} = \left|\frac{-b+\sqrt{b^2-3ac}}{3a} - \frac{-b-\sqrt{b^2-3ac}}{3a}\right|$$
$$\leq \left|\frac{-b+\sqrt{b^2-3ac}}{3a}\right| + \left|\frac{-b-\sqrt{b^2-3ac}}{3a}\right| \leq 2R$$

Therefore

$$\frac{\sqrt{b^2-3ac}}{3a} \leq R$$

and squaring both sides shows that

$$b^2 - 3ac \leq 9a^2R^2$$

or

$$c \geq \frac{b^2 - 9a^2R^2}{3a} \geq -\frac{9a^2R^2}{3a} = 3aR^2$$

We already know that $c < \frac{b^2}{3a}$, and since $|b| \leq 3aR$ it follows that

$$|c| \leq 3aR^2$$

Finally, let us consider the possible values of d. If $d > 7aR^3$, then since

$$at^3 \geq -aR^3 \quad bt^2 \geq -3aRt^2 \geq -3aR^3 \quad ct \geq -3aR^3$$

on $[-R, R]$ it follows that

$$p(t) = at^3 + bt^2 + ct + d > -7aR^3 + 7aR^3 > 0$$

on $[-R, R]$, which is a contradiction. Similarly, there is a contradiction if $d < -7aR^3$. Therefore $|d| \leq 7aR^3$.

Thus we have shown that there are $O(aR)$ choices for b, $O(aR^2)$ choices for c, and $O(aR^3)$ choices for d, hence $N_a(R) = O(a^3R^6)$. This proves that

$$\#\{(b, c, d) : |az^3 + bz^2 + cz + d| \in (aR, a(R+1)]\} \leq N_a(R+1) = O(a^3R^6) \tag{6.6}$$

Therefore

$$\sum_{\substack{b,c,d \in \mathbb{Z} \\ \mathrm{disc}(b,c,d)=\Delta}} |az^3 + bz^2 + cz + d|^{-k} \leq \sum_{\substack{b,c,d \in \mathbb{Z} \\ \mathrm{disc}(a,b,c,d)>0}} |az^3 + bz^2 + cz + d|^{-k}$$

$$\leq a^{-k} \sum_{R=1}^{\infty} \frac{N_a(R+1)}{R^k} \ll a^{3-k} \sum_{R=1}^{\infty} R^{k-6} \ll a^{3-k}$$

since $k \geq 8$, hence

$$\sum_{a=1}^{\infty} \sum_{\substack{a,b,c,d \in \mathbb{Z} \\ \operatorname{disc}(a,b,c,d)=\Delta}} |az^3 + bz^2 + cz + d|^{-k} \ll \sum_{a=1}^{\infty} a^{3-k} < \infty$$

6.3 Fourier expansion

Let $\Delta > 0$ be a cubic discriminant, and $k \geq 8$ an even integer. In the next few sections we will compute the Fourier coefficients of

$$h_{k,\Delta}(z) = \sum_{\substack{a,b,c,d \in \mathbb{Z} \\ \operatorname{disc}(a,b,c,d)=\Delta}} (az^3 + bz^2 + cz + d)^{-k} \tag{6.7}$$

where as usual z lies in the upper half plane $\mathbb{H}$ and

$$\operatorname{disc}(a,b,c,d) = -27a^2d^2 + 18abcd - 4ac^3 - 4b^3d + b^2c^2$$

is the discriminant of the cubic polynomial $az^3 + bz^2 + cz + d$.

Once again splitting up the sum according to the value of a, we obtain

$$h_{k,\Delta}(z) = \sum_{\substack{b,c,d \in \mathbb{Z} \\ \operatorname{disc}(0,b,c,d)=\Delta}} (bz^2 + cz + d)^{-k} + 2\sum_{a=1}^{\infty} \sum_{\substack{b,c,d \in \mathbb{Z} \\ \operatorname{disc}(a,b,c,d)=\Delta}} (az^3 + bz^2 + cz + d)^{-k} \tag{6.8}$$

For $r \in \mathbb{Z}$, the rth Fourier coefficient is then

$$\begin{aligned} c_r &= \int_i^{i+1} h_{k,\Delta}(z) e^{-2\pi i r z}\, dz \\ &= \sum_{\substack{b,c,d \in \mathbb{Z} \\ \operatorname{disc}(0,b,c,d)=\Delta}} \int_i^{i+1} (bz^2 + cz + d)^{-k} e^{-2\pi i r z}\, dz \\ &+2\sum_{a=1}^{\infty} \sum_{\substack{b,c,d \in \mathbb{Z} \\ \operatorname{disc}(a,b,c,d)=\Delta}} \int (az^3 + bz^2 + cz + d)^{-k} e^{-2\pi i r z}\, dz \end{aligned}$$

since the series converges locally uniformly.

The first term can be computed just as in section 4.6. So from now on, we will assume that $a \geq 1$.

6.4 Extending the integral

We now consider $a, \Delta > 0$ to be fixed. The first step is to extend the integral from a line segment to an entire line, which makes shifting contours easier. Let

$$\Gamma_\infty = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} : n \in \mathbb{Z} \right\}$$

Elements of Γ_∞ act on $\mathbb{H}$ by mapping z to $z+n$, which has the effect of sending az^3+bz^2+cz+d to $az^3 + b^*z^2 + c^*z + d^*$, where

$$b^* = b + 3an \tag{6.9}$$

$$c^* = c + 2bn + 3an^2 \tag{6.10}$$

$$d^* = d + cn + bn^2 + an^3 \tag{6.11}$$

We can now write (for a fixed $a > 0$)

$$\sum_{\substack{b,c,d \in \mathbb{Z} \\ \mathrm{disc}(a,b,c,d)=\Delta}} \int_i^{i+1} (az^3 + bz^2 + cz + d)^{-k} e^{-2\pi i r z}\, dz$$

$$= \sum_{\substack{(b,c,d) \bmod \Gamma_\infty \\ \mathrm{disc}(a,b,c,d)=\Delta}} \int_i^{i+1} \sum_{n \in \mathbb{Z}} [a(z+n)^3 + b(z+n)^2 + c(z+n) + d]^{-k} e^{-2\pi i r z}\, dz$$

$$= \sum_{\substack{(b,c,d) \bmod \Gamma_\infty \\ \mathrm{disc}(a,b,c,d)=\Delta}} \int_{i-\infty}^{i+\infty} [az^3 + bz^2 + cz + d]^{-k} e^{-2\pi i r z}\, dz$$

where $(b,c,d) \bmod \Gamma_\infty$ means that the sum runs over a set of representatives of the Γ_∞-equivalence classes of cubics $az^3 + bz^2 + cz + d$ for a fixed a.

Since $e^{-2\pi i r z} = e^{-2\pi i r x} e^{2\pi r y}$ and $y > 0$, if $r \leq 0$ then we can shift the contour toward $i\infty$, so the integral vanishes. So we are left with $r \geq 1$.

Now make two changes of variables in the integral. First, replace z with $z - \frac{b}{3a}$, so that $az^3 + bz^2 + cz + d$ becomes

$$a\Big(z - \frac{b}{3a}\Big)^3 + b\Big(z - \frac{b}{3a}\Big)^2 + c\Big(z - \frac{b}{3a}\Big) + d$$
$$= az^3 + \frac{3ac - b^2}{3a}z + \frac{2b^3 - 9abc + 27a^2d}{27a^2} = az^3 - \frac{\Delta_0}{3a}z + \frac{\Delta_1}{27a^2}$$

where

$$\Delta_0 = b^2 - 3ac \tag{6.12}$$

and

$$\Delta_1 = 2b^3 - 9abc + 27a^2d \tag{6.13}$$

Therefore after making the change of variables, we have

$$\int_{i-\infty}^{i+\infty} [az^3 + bz^2 + cz + d]^{-k} e^{-2\pi irz}\, dz = e^{\frac{2\pi irb}{3a}} \int_{i-\infty}^{i+\infty} \Big(az^3 - \frac{\Delta_0}{3a}z + \frac{\Delta_1}{27a^2}\Big)^{-k} e^{-2\pi irz}\, dz$$

Next, set $z = \frac{w}{a}$, so that $az^3 - \frac{\Delta_0}{3a}z + \frac{\Delta_1}{27a^2} = a^{-2}(w^3 - \frac{\Delta_0}{3}w + \frac{\Delta_1}{27})$, hence

$$e^{\frac{2\pi irb}{3a}} \int_{i-\infty}^{i+\infty} \Big(az^3 - \frac{\Delta_0}{3a}z + \frac{\Delta_1}{27a^2}\Big)^{-k} e^{-2\pi irz}\, dz = a^{2k-1} e^{\frac{2\pi irb}{3a}} \int_{i-\infty}^{i+\infty} \Big(w^3 - \frac{\Delta_0}{3}w + \frac{\Delta_1}{27}\Big)^{-k} e^{-\frac{2\pi irw}{a}}\, dw$$

Before attacking this last integral, let us find a convenient set of representatives for the Γ_∞-equivalence classes of cubics $az^3 + bz^2 + cz + d$.

6.5 Representatives for Γ_∞-equivalence classes

Throughout this section, Δ and a will be fixed positive integers.

Suppose that two cubic polynomials $az^3 + bz^2 + cz + d$ and $az^3 + b^*z^2 + c^*z + d^*$ having discriminant Δ are Γ_∞-equivalent. Then there is an integer n such that

$$az^3 + b^*z^2 + c^*z + d^* = a(z+n)^3 + b(z+n)^2 + c(z+n) + d$$

and from equations (6.9)-(6.11) we know that $b^* = b + 3an$, $c^* = c + 2bn + 3an^2$, and $d^* = d + cn + bn^2 + an^3$.

Now once again let $\Delta_0 = b^2 - 3ac$ and $\Delta_1 = 2b^3 - 9abc + 27a^2d$, and observe that

$$\Delta_1 = 2b(b^2 - 3ac) - 3a(bc - 9ad) = 2b\Delta_0 - 3a(bc - 9ad) \tag{6.14}$$

Let Δ_0^*, Δ_1^* be the corresponding polynomials in a, b^*, c^*, d^*. Then we have

$$\Delta_0^* = (b + 3an)^2 - 3a(c + 2bn + 3an^2) = b^2 + 6abn + 9a^2n^2 - 3ac - 6abn - 9a^2n^2$$

$$= b^2 - 3ac = \Delta_0$$

and

$$\Delta_1^* = 2b^*\Delta_0^* - 3a(b^*c^* - 9ad^*)$$

$$= 2(b + 3an)\Delta_0 - 3a[(b + 3an)(c + 2bn + 3an^2) - 9a(d + cn + bn^2 + an^3)]$$

Since

$$(b + 3an)(c + 2bn + 3an^2) - 9a(d + cn + bn^2 + an^3)$$

$$= bc + 2b^2n + 3abn^2 + 3acn + 6abn^2 + 9a^2n^3 - 9ad - 9acn - 9abn^2 - 9a^2n^3$$

$$= bc - 9ad + 2b^2n - 6acn = bc - 9ad + 2n\Delta_0$$

it follows that

$$\Delta_1^* = 2(b + 3an)\Delta_0 - 3a[(b + 3an)(c + 2bn + 3an^2) - 9a(d + cn + bn^2 + an^3)]$$

$$= 2b\Delta_0 + 6an\Delta_0 - 3a(bc - 9ad) - 6an\Delta_0$$

$$= 2b\Delta_0 - 3a(bc - 9ad) = \Delta_1$$

Therefore we have shown that if two cubic polynomials $az^3 + bz^2 + cz + d$ and $az^3 + b^*z^2 + c^*z + d^*$ are Γ_∞-equivalent, then

$$b^* \equiv b \mod 3a, \qquad \Delta_0^* = \Delta_0, \qquad \Delta_1^* = \Delta_1 \tag{6.15}$$

Conversely, suppose that two cubic polynomials $az^3 + bz^2 + cz + d$ and $az^3 + b^*z^2 + c^*z + d^*$ satisfy the three conditions in (6.15).

Then since $b^* \equiv b \pmod{3a}$, there is an integer n such that $b^* = b + 3an$. And because $\Delta_0^* = \Delta_0$, we have

$$c^* = \frac{(b^*)^2 - \Delta_0^*}{3a} = \frac{(b+3an)^2 - \Delta_0}{3a} = \frac{b^2 + 6abn + 9a^2n^2 - b^2 + 3ac}{3a}$$
$$= c + 2bn + 3an^2$$

and then since $\Delta_1^* = \Delta_1$, we get

$$d^* = \frac{\Delta_1^* - 2(b^*)^3 + 9a^*b^*c^*}{27a^2} = \frac{\Delta_1 - 2(b+3an)^3 + 9a(b+3an)(c+2bn+3an^2)}{27a^2}$$
$$= \frac{2b^3 - 9abc + 27a^2d - 2(b^3 + 9ab^2n + 27a^2bn^2 + 27a^3n^3) + 9a(b+3an)(c+2bn+3an^2)}{27a^2}$$
$$= d + cn + bn^2 + an^3$$

after some straightforward but tedious simplifications.

Thus we have shown that $b^* = b + 3an$, $c^* = c + 2bn + 3an^2$, and $d^* = d + cn + bn^2 + an^3$, which means that the cubic polynomials $az^3 + bz^2 + cz + d$ and $az^3 + b^*z^2 + c^*z + d^*$ are Γ_∞-equivalent.

Finally, let us consider which integers Δ_0, Δ_1 appear as above. From writing

$$3\Delta = 4(b^2 - 3ac)(c^2 - 3bd) - (bc - 9ad)^2 \tag{6.16}$$

it follows (after some algebra) that Δ_0, Δ_1 must satisfy

$$\Delta_1^2 - 4\Delta_0^3 = -27a^2\Delta \tag{6.17}$$

If we are given integers Δ_0, Δ_1 such that $\Delta_1^2 - 4\Delta_0^3 = -27a^2\Delta$, must they be Γ_∞-invariants of some cubic polynomial $az^3 + bz^2 + cz + d$? If we require that there is some $b \pmod{3a}$ for which

$$\Delta_0 \equiv b^2 \ (3a) \tag{6.18}$$

and

$$\Delta_1 \equiv b(3\Delta_0 - b^2) \ \ (27a^2) \tag{6.19}$$

then we can write $\Delta_0 = b^2 - 3ac$ for some integer c, and $\Delta_1 = 3b\Delta_0 - b^3 + 27a^2d = 2b^3 - 9abc + 27a^2d$ for some integer d, and then Δ_0, Δ_1 will be Γ_∞-invariants.

Hence the Fourier expansion (for $a \geq 1$) now has the form

$$\sum_{a=1}^{\infty} a^{2k-1} \sum_{b \bmod 3a} e^{\frac{2\pi i r b}{3a}} \sum_{\Delta_0 \equiv b^2\ (3a)} \sum_{\substack{\Delta_1 \equiv b(3\Delta_0 - b^2)\ (27a^2) \\ 4\Delta_0^3 - \Delta_1^2 = 27a^2\Delta}} \int_{i-\infty}^{i+\infty} \left(w^3 - \frac{\Delta_0}{3}w + \frac{\Delta_1}{27}\right)^{-k} e^{-\frac{2\pi i r w}{a}}\, dw \tag{6.20}$$

6.6 The residue computation

Having now chosen a convenient set of representatives for the Γ_∞-equivalence classes, we now want to evaluate the integral

$$a^{2k-1} e^{\frac{2\pi i r b}{3a}} \int_{i-\infty}^{i+\infty} \left(w^3 - \frac{\Delta_0}{3}w + \frac{\Delta_1}{27}\right)^{-k} e^{-\frac{2\pi i r w}{a}}\, dw \tag{6.21}$$

While we are ultimately interested in even integers $k \geq 8$, the integral above converges even when $k = 1$ since the cubic polynomial has real roots. So we will start by considering the integral

$$a e^{\frac{2\pi i r b}{3a}} \int_{i-\infty}^{i+\infty} \left(w^3 - \frac{\Delta_0}{3}w + \frac{\Delta_1}{27}\right)^{-1} e^{-\frac{2\pi i r w}{a}}\, dw$$

and later repeatedly differentiate this integral with respect to Δ_1.

Begin by writing

$$w^3 - \frac{\Delta_0}{3}w + \frac{\Delta_1}{27} = (w - r_1)(w - r_2)(w - r_3) \tag{6.22}$$

The discriminant of this cubic is

$$-4\Big(-\frac{\Delta_0}{3}\Big)^3 - 27\Big(\frac{\Delta_1}{27}\Big)^2 = \frac{4\Delta_0^3 - \Delta_1^2}{27} = a^2\Delta$$

so order the roots r_1, r_2, r_3 such that

$$(r_1 - r_2)(r_1 - r_3)(r_2 - r_3) = a\sqrt{\Delta} \tag{6.23}$$

If $w = u + iv$ with $v > 0$, then $e^{-\frac{2\pi i r w}{a}} = e^{-\frac{2\pi i r u}{a}} e^{\frac{2\pi r v}{a}}$, and since $r, a > 0$ we can shift the contour of integration to $-i\infty$ to obtain

$$a e^{\frac{2\pi i r b}{3a}} \int_{i-\infty}^{i+\infty} (w^3 - \frac{\Delta_0}{3}w + \frac{\Delta_1}{27})^{-1} e^{-\frac{2\pi i r w}{a}}\, dw$$

$$= -ae^{\frac{2\pi irb}{3a}} 2\pi i \sum_{j=1}^{3} \mathrm{Res}_{w=r_j} \frac{e^{-\frac{2\pi irw}{a}}}{(w-r_1)(w-r_2)(w-r_3)}$$

$$= -2\pi iae^{\frac{2\pi irb}{3a}} \Big[\frac{e^{-\frac{2\pi irr_1}{a}}}{(r_1-r_2)(r_1-r_3)} + \frac{e^{-\frac{2\pi irr_2}{a}}}{(r_2-r_1)(r_2-r_3)} + \frac{e^{-\frac{2\pi irr_3}{a}}}{(r_3-r_1)(r_3-r_2)}\Big]$$

$$= -2\pi iae^{\frac{2\pi irb}{3a}} \frac{(r_2-r_3)e^{-\frac{2\pi irr_1}{a}} - (r_1-r_3)e^{-\frac{2\pi irr_2}{a}} + (r_1-r_2)e^{-\frac{2\pi irr_3}{a}}}{(r_1-r_2)(r_1-r_3)(r_2-r_3)}$$

To handle the numerator, expand the three exponentials out in a power series:

$$e^{-\frac{2\pi irr_j}{a}} = \sum_{m=0}^{\infty} \frac{(-\frac{2\pi ir}{a})^m}{m!} r_j^m$$

Collecting terms containing $\frac{(-\frac{2\pi ir}{a})^m}{m!}$ leads to

$$(r_2-r_3)e^{-\frac{2\pi irr_1}{a}} - (r_1-r_3)e^{-\frac{2\pi irr_2}{a}} + (r_1-r_2)e^{-\frac{2\pi irr_3}{a}}$$

$$= \sum_{m=0}^{\infty} \frac{(-\frac{2\pi ir}{a})^m}{m!} [r_1^m(r_2-r_3) - r_2^m(r_1-r_3) + r_3^m(r_1-r_2)]$$

Next, observe that

$$[r_1^m(r_2-r_3) - r_2^m(r_1-r_3) + r_3^m(r_1-r_2) = \begin{vmatrix} r_1^m & r_1 & 1 \\ r_2^m & r_2 & 1 \\ r_3^m & r_3 & 1 \end{vmatrix} \tag{6.24}$$

and similarly

$$(r_1-r_2)(r_1-r_3)(r_2-r_3) = \begin{vmatrix} r_1^2 & r_1 & 1 \\ r_2^2 & r_2 & 1 \\ r_3^2 & r_3 & 1 \end{vmatrix} \tag{6.25}$$

Therefore the quotient

$$s_m(r_1,r_2,r_3) = \frac{\begin{vmatrix} r_1^m & r_1 & 1 \\ r_2^m & r_2 & 1 \\ r_3^m & r_3 & 1 \end{vmatrix}}{\begin{vmatrix} r_1^2 & r_1 & 1 \\ r_2^2 & r_2 & 1 \\ r_3^2 & r_3 & 1 \end{vmatrix}} \tag{6.26}$$

is a symmetric polynomial in r_1, r_2, r_3 (called a Schur polynomial), and we have

$$ae^{\frac{2\pi irb}{3a}} \int_{i-\infty}^{i+\infty} (w^3 - \frac{\Delta_0}{3}w + \frac{\Delta_1}{27})^{-1} e^{-\frac{2\pi irw}{a}}\, dw$$

$$= -2\pi iae^{\frac{2\pi irb}{3a}} \sum_{m=0}^{\infty} \frac{(-\frac{2\pi ir}{a})^m}{m!} s_m(r_1, r_2, r_3) \tag{6.27}$$

This representation has the disadvantage of being expressed in the roots of the polynomial, while we would like a formula in terms of the coefficients of the polynomial. But as we will see in the next section, this can be fixed.

6.7 Schur polynomials

Given variables $x_1, \dots, x_n$ and a partition $\lambda = (\lambda_1, \dots, \lambda_n)$ of an integer m with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$, define the Schur polynomial

$$s_\lambda(x_1, \dots, x_n) = \frac{\begin{vmatrix} x_1^{\lambda_1+n-1} & x_1^{\lambda_2+n-2} & \cdots & x_1^{\lambda_n} \\ x_2^{\lambda_1+n-1} & x_2^{\lambda_2+n-2} & \cdots & x_2^{\lambda_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\lambda_1+n-1} & x_n^{\lambda_2+n-2} & \cdots & x_n^{\lambda_n} \end{vmatrix}}{\begin{vmatrix} x_1^{n-1} & x_1^{n-2} & \cdots & 1 \\ x_2^{n-1} & x_2^{n-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{n-1} & x_n^{n-2} & \cdots & 1 \end{vmatrix}}$$

$s_\lambda(x_1, \dots, x_n)$ is a symmetric polynomial in the x_i because it is the quotient of two alternating polynomials, and hence can be expressed in terms of the elementary symmetric polynomials in the x_i. The identity relating the s_λ to the elementary symmetric polynomials is called the second Jacobi-Trudi formula, and works as follows:

Given a partition λ as above, define a conjugate partition λ' by reflecting the Ferrers diagram of λ about its diagonal. Let ℓ be the length of λ', i.e. the number of non-zero terms in λ'. The Jacobi-Trudi formula (see Appendix A of [FH91]) states that

$$s_\lambda(x_1,\dots,x_n) = \begin{vmatrix} e_{\lambda'_1} & e_{\lambda'_1+1} & \cdots & e_{\lambda'_1+\ell-1} \\ e_{\lambda'_2-1} & e_{\lambda'_2} & \cdots & e_{\lambda'_2+\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ e_{\lambda'_\ell-\ell+1} & e_{\lambda'_\ell-\ell+2} & \cdots & e_{\lambda'_\ell} \end{vmatrix}$$

Here e_j is the jth elementary symmetric polynomial in $x_1,\dots,x_n$, with the convention $e_j = 0$ if $j < 0$ or $j > n$.

Specializing to the situation of the previous section, we have $n = 3$ and $\lambda = (m,0,0)$, so $\lambda' = (1,1,\dots,1)$ (m times). Therefore

$$s_{(m,0,0)}(r_1,r_2,r_3) = \begin{vmatrix} 0 & e_2 & e_3 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & e_2 & e_3 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & e_2 & e_3 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{vmatrix}$$

using the fact that $e_1(r_1,r_2,r_3) = 0$. Since $e_2(r_1,r_2,r_3) = -\frac{\Delta_0}{3}$ and $e_3(r_1,r_2,r_3) = -\frac{\Delta_1}{27}$, we have expressed $s_{(m,0,0)}$ as a polynomial in Δ_0 and Δ_1.

Moreover, the polynomials $s_{(m,0,0)}$ obey a recurrence relation: expanding the above determinant along the first row, we obtain

$$s_{(m,0,0)} = -e_2M_1 + e_3M_2 \tag{6.28}$$

where M_1, M_2 are $(m-1)\times(m-1)$ minors of the matrix. The first column of M_1 is $\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$,

and expanding along this column we obtain the determinant defining $s_{(m-2,0,0)}$.

Similarly, the first two columns of M_2 are $\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}$, so expanding along these columns yields the determinant defining $s_{(m-3,0,0)}$. Therefore

$$s_{(m,0,0)} = -e_2 s_{(m-2,0,0)} + e_3 s_{(m-3,0,0)} \tag{6.29}$$

for $m \geq 3$, which along with $s_{(0,0,0)} = 1$, $s_{(1,0,0)} = 0$, and $s_{(2,0,0)} = -e_2$ allows us (in principle) to compute $s_{(m,0,0)}$ for all m.

6.8 Putting it all together

Using the result of the previous section, we can write

$$ae^{\frac{2\pi irb}{3a}} \int_{i-\infty}^{i+\infty} \Big(w^3 - \frac{\Delta_0}{3}w + \frac{\Delta_1}{27}\Big)^{-1} e^{-\frac{2\pi irw}{a}}\ dw$$

$$= -2\pi iae^{\frac{2\pi irb}{3a}} \sum_{m=0}^{\infty} \frac{(-\frac{2\pi ir}{a})^m}{m!} p_m(\Delta_0, \Delta_1)$$

where $p_m(\Delta_0, \Delta_1)$ is the polynomial in Δ_0 and Δ_1 determined by $s_m(r_1, r_2, r_3)$. Therefore

$$a^{2k-1}e^{\frac{2\pi irb}{3a}} \int_{i-\infty}^{i+\infty} (w^3 - \frac{\Delta_0}{3}w + \frac{\Delta_1}{27})^{-k} e^{-\frac{2\pi irw}{a}}\ dw$$

$$= \frac{a^{2k-2}27^{k-1}}{(k-1)!} \frac{d^{k-1}}{d\Delta_1^{k-1}} \Big[ae^{\frac{2\pi irb}{3a}} \int_{i-\infty}^{i+\infty} (w^3 - \frac{\Delta_0}{3}w + \frac{\Delta_1}{27})^{-1} e^{-\frac{2\pi irw}{a}}\ dw\Big]$$

$$= \frac{2\pi ia^{2k-1}e^{\frac{2\pi irb}{3a}}27^{k-1}}{(k-1)!} \frac{d^{k-1}}{d\Delta_1^{k-1}} \sum_{m=0}^{\infty} \frac{(-\frac{2\pi ir}{a})^m}{m!} p_m(\Delta_0, \Delta_1)$$

If we denote this last expression by

$$e^{\frac{2\pi irb}{3a}} j_k(r, a, \Delta_0, \Delta_1) \tag{6.30}$$

then the portion of the Fourier expansion coming from the terms with $a \geq 1$ has the form

$$\sum_{a=1}^{\infty} \sum_{b \bmod 3a} e^{\frac{2\pi irb}{3a}} \sum_{\Delta_0 \equiv b^2\ (3a)} \sum_{\substack{\Delta_1 \equiv b(3\Delta_0 - b^2)\ (27a^2) \\ 4\Delta_0^3 - \Delta_1^2 = 27a^2\Delta}} j_k(r, a, \Delta_0, \Delta_1) \tag{6.31}$$

The order of the sums can be changed to isolate the trigonometric sum, leading to

$$\sum_{a=1}^{\infty} \sum_{\Delta_0 \in \mathbb{Z}} \sum_{\substack{\Delta_1 \in \mathbb{Z} \\ 4\Delta_0^3 - \Delta_1^2 = 27a^2\Delta}} j_k(r, a, \Delta_0, \Delta_1) \sum_{\substack{b \bmod 3a \\ \Delta_0 \equiv b^2\ (3a) \\ \Delta_1 \equiv b(3\Delta_0 - b^2)\ (27a^2)}} e^{\frac{2\pi irb}{3a}} \tag{6.32}$$

6.9 Another approach to the "Bessel" function

A different expression for the function $j_k(r, a, \Delta_0, \Delta_1)$ can be obtained using Cardano's formula for the roots of a cubic. Given $az^3 + bz^2 + cz + d$ with $a > 0$, write

$$az^3 + bz^2 + cz + d = a(z - r_1)(z - r_2)(z - r_3)$$

where

$$r_1 = -\frac{1}{3a}\left(b + C + \frac{\Delta_0}{C}\right) \tag{6.33}$$

$$r_2 = -\frac{1}{3a}\left(b + \omega C + \frac{\Delta_0}{\omega C}\right) \tag{6.34}$$

$$r_3 = -\frac{1}{3a}\left(b + \omega^2 C + \frac{\Delta_0}{\omega^2 C}\right) \tag{6.35}$$

Here $\omega = e^{\frac{2\pi i}{3}}$ is a primitive cube root of unity, Δ_0 and Δ_1 are defined as above, and

$$C = \sqrt[3]{\frac{\Delta_1 + \sqrt{-27a^2\Delta}}{2}} \tag{6.36}$$

There are three choices for the cube root in the definition of C, but the particular choice that is made is not especially important, as all three cube roots appear in equations (6.33)-(6.35) above. For simplicity, we will assume that the cube root is chosen so that

$$a^2(r_1 - r_2)(r_1 - r_3)(r_2 - r_3) = \sqrt{\Delta} \tag{6.37}$$

Now shifting contours as in section 6, we obtain

$$\int_{i-\infty}^{i+\infty} (az^3 + bz^2 + cz + d)^{-1} e^{-2\pi irz}\, dz = -\frac{2\pi i}{a} \sum_{j=1}^{3} \mathrm{Res}_{z=r_j} \frac{e^{-2\pi irz}}{(z - r_1)(z - r_2)(z - r_3)}$$

$$= -\frac{2\pi i}{a}\Big[\frac{e^{-2\pi irr_1}}{(r_1-r_2)(r_1-r_3)} + \frac{e^{-2\pi irr_2}}{(r_2-r_1)(r_2-r_3)} + \frac{e^{-2\pi irr_3}}{(r_3-r_1)(r_3-r_2)}\Big]$$

$$= -\frac{2\pi i}{a(r_1-r_2)(r_1-r_3)(r_2-r_3)}\Big[(r_2-r_3)e^{-2\pi irr_1} - (r_1-r_3)e^{-2\pi irr_2} + (r_1-r_2)e^{-2\pi irr_3}\Big]$$

$$= -\frac{2\pi ia}{\sqrt{\Delta}}\Big[(r_2-r_3)e^{-2\pi irr_1} - (r_1-r_3)e^{-2\pi irr_2} + (r_1-r_2)e^{-2\pi irr_3}\Big]$$

Now recall from [WW96] the generating function for Bessel functions of integer order: if $z, t \in \mathbb{C}$ with $t \neq 0$, then

$$e^{\frac{z}{2}(t-\frac{1}{t})} = \sum_{m=-\infty}^{\infty} J_m(z)t^m \tag{6.38}$$

Using equations (6.33)-(6.35) then yields

$$e^{-2\pi irr_1} = e^{\frac{2\pi irb}{3a}} e^{\frac{2\pi r\sqrt{\Delta_0}}{3a}(\frac{iC}{\sqrt{\Delta_0}} - \frac{\sqrt{\Delta_0}}{iC})} = e^{\frac{2\pi irb}{3a}} \sum_{m=-\infty}^{\infty} J_m\Big(\frac{4\pi r\sqrt{\Delta_0}}{3a}\Big)\Big(\frac{iC}{\sqrt{\Delta_0}}\Big)^m \tag{6.39}$$

and similarly

$$e^{-2\pi irr_2} = e^{\frac{2\pi irb}{3a}} e^{\frac{2\pi r\sqrt{\Delta_0}}{3a}(\frac{i\omega C}{\sqrt{\Delta_0}} - \frac{\sqrt{\Delta_0}}{i\omega C})} = e^{\frac{2\pi irb}{3a}} \sum_{m=-\infty}^{\infty} J_m\Big(\frac{4\pi r\sqrt{\Delta_0}}{3a}\Big)\Big(\frac{i\omega C}{\sqrt{\Delta_0}}\Big)^m \tag{6.40}$$

and

$$e^{-2\pi irr_3} = e^{\frac{2\pi irb}{3a}} e^{\frac{2\pi r\sqrt{\Delta_0}}{3a}(\frac{i\omega^2 C}{\sqrt{\Delta_0}} - \frac{\sqrt{\Delta_0}}{i\omega^2 C})} = e^{\frac{2\pi irb}{3a}} \sum_{m=-\infty}^{\infty} J_m\Big(\frac{4\pi r\sqrt{\Delta_0}}{3a}\Big)\Big(\frac{i\omega^2 C}{\sqrt{\Delta_0}}\Big)^m \tag{6.41}$$

Making another use of equations (6.33)-(6.35), it follows that

$$-\frac{2\pi ia}{\sqrt{\Delta}}\Big[(r_2-r_3)e^{-2\pi irr_1} - (r_1-r_3)e^{-2\pi irr_2} + (r_1-r_2)e^{-2\pi irr_3}\Big]$$

$$= -\frac{2\pi i}{3\sqrt{\Delta}} e^{\frac{2\pi irb}{3a}}\Big[([\omega^2-\omega]C + [\omega-\omega^2]\frac{\Delta_0}{C}) \sum_{m=-\infty}^{\infty} J_m\Big(\frac{4\pi r\sqrt{\Delta_0}}{3a}\Big)\Big(\frac{iC}{\sqrt{\Delta_0}}\Big)^m$$

$$+([1-\omega^2]C + [1-\omega]\frac{\Delta_0}{C}) \sum_{m=-\infty}^{\infty} J_m\Big(\frac{4\pi r\sqrt{\Delta_0}}{3a}\Big)\Big(\frac{i\omega C}{\sqrt{\Delta_0}}\Big)^m$$

$$+([\omega-1]C + [\omega^2-1]\frac{\Delta_0}{C}) \sum_{m=-\infty}^{\infty} J_m\Big(\frac{4\pi r\sqrt{\Delta_0}}{3a}\Big)\Big(\frac{i\omega^2 C}{\sqrt{\Delta_0}}\Big)^m\Big]$$

The coefficient of J_m inside the brackets is

$$\frac{i^m C^{m+1}}{\Delta_0^{\frac{m}{2}}}\left[\omega^2-\omega+\omega^m-\omega^{m+2}+\omega^{2m+1}-\omega^{2m}\right]+\frac{i^m C^{m-1}}{\Delta_0^{\frac{m}{2}-1}}\left[\omega-\omega^2+\omega^m-\omega^{m+1}+\omega^{2m+2}-\omega^{2m}\right]$$

The expression inside the first set of brackets is zero unless $m \equiv 2 \pmod 3$, in which case it is equal to $3(\omega^2-\omega)$. Similarly, the expression inside the second set of brackets is zero unless $m \equiv 1 \pmod 3$, in which case it is equal to $3(\omega-\omega^2)$.

Therefore setting $m = 3v+2$ or $m = 3v+1$, we finally obtain

$$-\frac{2\pi i}{\sqrt{\Delta}}e^{\frac{2\pi i r b}{3a}}\left[(\omega^2-\omega)\sum_{v=-\infty}^{\infty}\frac{i^{3v+2}C^{3(v+1)}}{\Delta_0^{\frac{3v}{2}+1}}J_{3v+2}\left(\frac{4\pi r\sqrt{\Delta_0}}{3a}\right)+(\omega-\omega^2)\sum_{v=-\infty}^{\infty}\frac{i^{3v+1}C^{3v}}{\Delta_0^{\frac{3v-1}{2}}}J_{3v+1}\left(\frac{4\pi r\sqrt{\Delta_0}}{3a}\right)\right]$$

Notice that the choice of cube root in the definition of C is now irrelevant.

6.10 The case $\Delta = 0$

Finally, let us consider $h_{k,0}(z)$. As with the $m = 0$ case in the previous chapter, the definition is slightly different: define

$$h_{k,0}(z) = \sideset{}{'}\sum_{\substack{a,b,c,d\in\mathbb{Z}\\ \mathrm{disc}(a,b,c,d)=0\\ b^2-3ac=0=c^2-3bd}} (az^3+bz^2+cz+d)^{-k} \tag{6.42}$$

where as always the sum omits the term $a = b = c = d = 0$.

The reasoning for the extra conditions is similar to that of the previous chapter: if $\mathrm{disc}(a,b,c,d) > 0$ then we have seen that necessarily $b^2-3ac > 0$, and since

$$3\cdot\mathrm{disc}(a,b,c,d) = 4(b^2-3ac)(c^2-3bd)-(bc-9ad)^2 \tag{6.43}$$

it follows that in this case $c^2-3bd > 0$ as well. In the case $\mathrm{disc}(a,b,c,d) = 0$, however, it is convenient to include these extra conditions.

We will show that $h_{k,0}(z)$ is a modular form by showing that it is a multiple of the Eisenstein series of weight $3k$ for $\mathrm{SL}_2(\mathbb{Z})$. First observe that the conditions in the definition of $h_{k,0}(z)$ are homogeneous in a, b, c, d, and so we can write

$$h_{k,0}(z) = \zeta(k)\sum_{\substack{\gcd(a,b,c,d)=1\\ \mathrm{disc}(a,b,c,d)=0\\ b^2-3ac=0=c^2-3bd}} (az^3+bz^2+cz+d)^{-k} \tag{6.44}$$

Next, the conditions $\gcd(a, b, c, d) = 1$, $b^2 = 3ac$, and $c^2 = 3bd$ imply that $\gcd(a, d) = 1$ and (from (6.43)) that $bc = 9ad$. Since b and c are divisible by 3, write $b = 3b'$ and $c = 3c'$, so that now

$$(b')^2 = ac' \qquad (c')^2 = b'd \qquad b'c' = ad \tag{6.45}$$

Let p be a prime dividing a. It then follows that p divides b', hence divides c', and therefore that $p^2|b'$ since p does not divide d. It then follows from $b'c' = ad$ that $p^3|a$, and so a is a perfect cube. Similarly, one can show that d is a perfect cube.

Therefore we can write $a = m^3$ and $b = n^3$ for integers m, n with $\gcd(m, n) = 1$, and then it is easy to check that $b' = m^2n$ and $c' = mn^2$. Therefore (6.44) becomes

$$h_{k,0}(z) = \zeta(k) \sum_{\gcd(m,n)=1} (mz + n)^{-3k} = \frac{\zeta(k)}{\zeta(3k)} G_{3k}(z) \tag{6.46}$$

where

$$G_{3k}(z) = \sum_{m,n\in\mathbb{Z}}{}' (mz + n)^{-3k}$$

is the Eisenstein series of weight $3k$ for $\mathrm{SL}_2(\mathbb{Z})$.

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